Unit #5 : Exponential Functions (10 days + 1 jazz day + 1 summative evaluation day)

BIG Ideas:

Students will:

- **Collect primary data and investigate secondary data that can be modelled as exponential growth/decay functions**
- **Make connections between numeric, graphical and algebraic representations of exponential functions**
- **Identify key features of the graphs of exponential functions (e.g., domain, range, y-intercept, horizontal asymptote, increasing and decreasing)**
- **Apply an understanding of domain and range to a variety of exponential models**
- **Solve real-world applications using given graphs or equations of exponential functions**
- **Simplify and evaluate numerical expressions involving exponents**

DAY	Lesson Title & Description	2P	2D	Expectations		Teaching/Assessment Notes and Curriculum Sample Problems
1	Piles of Homework • Distinguish exponential functions from linear and quadratic by examining tables of values and graphs ***Lesson Included***	N	N	EF1.06 ✓	distinguish exponential functions from linear and quadratic functions by making comparisons in a variety of ways (e.g., comparing rates of change using finite differences in tables of values; identifying a constant ratio in a table of values; inspecting graphs; comparing equations), within the same context when possible (e.g., simple interest and compound interest; population growth)	***Sample problem:*** Explain in a variety of ways how you can distinguish the exponential function $f(x) = 2^x$ from the quadratic function $f(x) = x^2$ and the linear function $f(x) = 2x$.
2	Investigating Exponential Growth • Collect data that can be modelled as exponential growth functions through investigation and from secondary sources • Make connections to First Differences and constant ratios ***Lesson Included***	N	N	EF2.01 ✓	collect data that can be modelled as an exponential function, through investigation with and without technology, from primary sources, using a variety of tools (e.g., concrete materials such as number cubes, coins; measurement tools such as electronic probes), or from secondary sources (e.g., websites such as Statistics Canada, E-STAT), and graph the data	***Sample problem:*** Collect data and graph the cooling curve representing the relationship between temperature and time for hot water cooling in a porcelain mug. Predict the shape of the cooling curve when hot water cools in an insulated mug. Test your prediction.
3	Investigating Exponential Decay • Collect data that can be modelled as exponential decay functions • Make connections to First Differences and constant ratios ***Lesson Included***					

4	Investigating The Graphs of Exponential Functions – Day 1 - Graph exponential functions in the form $y = ab^x$ where $b>0$ and $a=1$ - Identify key features (y-intercept, increasing or decreasing, domain and range, horizontal asymptotes, constant ratio) ***Lesson Included***	N	N	EF1.03 ✓ EF1.04 ✓	graph, with and without technology, an exponential relation, given its equation in the form $y = a^x$ ($a > 0, a \neq 1$), define this relation as the function $f(x) = a^x$, and explain why it is a function; determine, through investigation, and describe key properties relating to domain and range, intercepts, increasing/decreasing intervals, and asymptotes (e.g., the domain is the set of real numbers; the range is the set of positive real numbers; the function either increases or decreases throughout its domain) for exponential functions represented in a variety of ways [e.g., tables of values, mapping diagrams, graphs, equations of the form $f(x) = a^x$ ($a > 0, a \neq 1$), function machines]	***Sample problem:*** Graph $f(x) = 2^x$, $g(x) = 3^x$, and $h(x) = 0.5^x$ on the same set of axes. Make comparisons between the graphs, and explain the relationship between the y-intercepts. 🖩
5	Investigating The Graphs of Exponential Functions – Day 2 - Graph exponential functions in the form $y = ab^x$ where $b>0$ and $a>1$ - Identify key features (y-intercept, increasing or decreasing, domain and range, horizontal asymptotes, constant ratio) ***Lesson Included***					
6	Domain and Range in Real World Applications - Identify exponential functions that arise from real world applications involving growth and decay - Determine reasonable restrictions on the domain and range ***Lesson Included***	N	N	EF2.02 ✓	identify exponential functions, including those that arise from real-world applications involving growth and decay (e.g., radioactive decay, population growth, cooling rates, pressure in a leaking tire), given various representations (i.e., tables of values, graphs, equations), and explain any restrictions that the context places on the domain and range (e.g., ambient temperature limits the range for a cooling curve);	🖩

7	How an Infectious Disease can Spread • Simulate the spread of an infectious disease and analyze the results • Determine restrictions that must be placed on the domain and range in order to apply an exponential model ***Lesson Included***	N	N	EF2.02 ✓	identify exponential functions, including those that arise from real-world applications involving growth and decay (e.g., radioactive decay, population growth, cooling rates, pressure in a leaking tire), given various representations (i.e., tables of values, graphs, equations), and explain any restrictions that the context places on the domain and range (e.g., ambient temperature limits the range for a cooling curve);	This activity requires advanced preparation. Full Teacher Notes are provided in BLM 5.7.2 🖩
8,9	Developing and Applying Exponent Laws • Investigate to develop exponent laws for multiplying and dividing numerical expressions involving exponents and for finding the power of a power . • Investigate to find the value of a power with a rational exponent (e.g., use a graphing calculator to find the value for $4^{\frac{1}{2}}$ or $27^{\frac{1}{3}}$ by entering an exponential function with the given base and then using TRACE.) • Evaluate numerical expressions with rational bases and integer/rational exponents. • Note: Students only work with numerical expressions	N	C	EF1.05 ✓	determine, through investigation (e.g., by patterning with and without a calculator), the exponent rules for multiplying and dividing numerical expressions involving exponents [e.g.,$(½)^3$ x $(½)^2$], and the exponent rule for simplifying numerical expressions involving a power of a power [e.g.,$(5^3)^2$], and use the rules to simplify numerical expressions containing integer exponents [e.g., $(2^3)(2^5) = 2^8$];	Note: Students don't actually solve exponential equations in this course so the main use of these exponent rules would likely be to help develop an understanding of rational exponents (see sample problem below) and to understand the compound interest formula
		N	N	EF1.01 ☑	determine, through investigation using a variety of tools (e.g., calculator, paper and pencil, graphing technology) and strategies (e.g., patterning; finding values from a graph; interpreting the exponent laws), the value of a power with a rational exponent (i.e., $x^{\frac{m}{n}}$, where $x > 0$ and m and n are integers)	***Sample problem:*** The exponent laws suggest that $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^1$. What value would you assign to $4^{\frac{1}{2}}$? What value would you assign to $27^{\frac{1}{3}}$? Explain your reasoning. Extend your reasoning to make a generalization about the meaning of $x^{\frac{1}{n}}$, where $x > 0$ and n is a natural number. **<u>Suggestion</u>**: Teachers may want to have students explore on sketchpad or with a graphing calculator. Students can graph $y = 4^x$ and then examine the y-value when x = ½ and then graph $y = 9^x$ and examine the y-value when x= ½ and so on. 🖩 and/or 💻
		N	C	EF1.02 ☑	evaluate, with and without technology, numerical expressions containing integer and rational exponents and rational bases [e.g., 2^{-3}, $(-6)^3$, $4^{\frac{1}{2}}$, 1.01^{120}];	

10	Using Graphical and Algebraic Models • Students will solve problems using given graphs or equations of exponential functions • Help students make connections between the algebraic model of the exponential function and the real-world application (i.e. help students understand the meanings of *a* and *b* in the context of the problem) • Note: Students are not required to generate the equation on their own, but should be encouraged to explain the parameters in the context of the problem. ***Lesson Included***	N	N	EF2.03 ✓	solve problems using given graphs or equations of exponential functions arising from a variety of real-world applications (e.g., radioactive decay, population growth, height of a bouncing ball, compound interest) by interpreting the graphs or by substituting values for the exponent into the equations	***Sample problem:*** The temperature of a cooling liquid over time can be modelled by the exponential function $T(x)=60\left(\frac{1}{2}\right)^{\frac{x}{30}}+20$, where $T(x)$ is the temperature, in degrees Celsius, and x is the elapsed time, in minutes. Graph the function and determine how long it takes for the temperature to reach 28ºC. 🖩
11	Review Day (Jazz Day)					
12	Summative Evaluation					

Unit 5 : Day 1 : Piles of Homework	**Grade 11 U/C**

Minds On: 20 Action: 35 Consolidate:20 Total=75 min	**Description/Learning Goals** • Distinguish exponential functions from linear and quadratic functions in a variety of ways (eg. Comparing rates of change using finite differences; inspecting graphs, identifying a constant ratio in a table)	**Materials** • BLM 5.1.1 • BLM 5.1.2 • BLM 5.1.3 • Bundle of photocopy paper • 6 file folders • post-it notes • 25 paper clips

				Assessment Opportunities
🟩	**Minds On…**	**Individual → Activating Prior Knowledge** Paper folding – review of exponents Students receive a piece of paper. Have students fold paper in half and record the number of layers. Repeat this process and have students record the number of folds and the number of layers created by the fold. Connections can then be made to powers of 2. **Whole class → Guided Exploration** Write the list as a sequence to look for a pattern (constant ratio). Write some other patterns that use constant ratio on the board and ask students to complete the patterns and identify how they filled in the missing parts.		For the Action! component, have paper available for learners to use for representation.
🟥	**Action!**	**Jigsaw → Investigation** Students are first organized into *home* groups of 3. Each member will then be assigned a number. The home groups will then divide into 6 groups. There should be two groups of 1s, 2s, and 3s. In each group, one student is selected to play the role of *Teacher*. Provide each group a set of instructions from BLM 5.1.1. The *Teacher* in the group receives a file folder that has groups of paper clipped together. Each set of clipped papers represents the homework for one day. The *Teacher* hands out the homework one day at a time and the students will record how many pages of homework they have received in total. The students record the work on BLM 5.1.2. After recording the data, the students return to the *home* group to share the data. In the home group, the students complete the remainder of BLM 5.1.2 **Mathematical Process Focus**: **Problem solving** (students decide on which model is being shown), **connecting** (students are using skills developed for selecting a type of model), **representing** (students represent a situation with a model).		For the small group activity, it is recommended to try a jig-saw form of cooperative learning. Additional information on the jig-saw form can be found in *Think Literacy. Cross-curricular Approaches Grades 7- 12. 2003. pg. 170.*
🟦	**Consolidate Debrief**	**Whole Class → Guided discussion** Discuss with the class the questions from BLM 5.1.1 about how much homework they would have received in 30 days from each class. **Whole Class → Note taking** Guiding question: *Can first and second differences be used to determine if a model is an exponential model?* The teacher guides the class through developing a note and introduction of key term *constant ratio*. Begin to build the Exponential FRAME with students (BLM 5.1.3).		**Literacy Strategy**: Students can begin to add information to the **Exponential FRAME Model (BLM 5.1.3).** **(F** unction, **R** epresentation**, A** nd, **M** odel, **Example)** The FRAME can be extended throughout the unit.
	Concept Practice	**Home Activity or Further Classroom Consolidation** The teacher provides students with additional tables of values that include linear / quadratic/ exponential values and students are to try and determine which model the table of values represents.		

5.1.1 Piles of Homework

Photocopy this sheet and cut into sections to give to groups.

Scenario: You have discovered that three of your teachers give out homework in different ways. How much homework will you have on your homework pile?

Class A
How much homework do you get from your teacher?
Record how many pieces of paper you have **in total** at the end of each day in your table.

After 30 days, how many pieces of paper will you have been given?

Scenario: You have discovered that three of your teachers give out homework in different ways. How much homework will you have on your homework pile?

Class B
How much homework do you get from your teacher?
Record how many pieces of paper you have **in total** at the end of each day in your table.

After 30 days, how many pieces of paper will you have been given?

Scenario: You have discovered that three of your teachers give out homework in different ways. How much homework will you have on your homework pile?

Class C
How much homework do you get from your teacher?
Record how many pieces of paper you have **in total** at the end of each day in your table.

Note: Is there a connection between how many pieces of paper you have on your pile and how many pieces of paper you get the next day?

After 30 days, how many pieces of paper will you have been given?

5.1.1 Piles of Homework (continued)

Teacher notes.

How to organize the paper for Class A

Day 1	1 sheet of paper
Day 2	2 sheets of paper
Day 3	2 sheets of paper
Day 4	2 sheets of paper
Day 5	2 sheets of paper

How to organize the paper for Class B

Day 1	1 sheet of paper
Day 2	3 sheets of paper
Day 3	5 sheets of paper
Day 4	7 sheets of paper
Day 5	9 sheets of paper

How to organize the paper for Class C

Day 1	1 sheet of paper
Day 2	2 sheets of paper
Day 3	6 sheets of paper
Day 4	18 sheets of paper
Day 5	54 sheets of paper

It is recommended you use a post-it note to stick on the top of each pile in order for the students to know which day of homework they are receiving.

5.1.2 Piles of Homework

Scenario: You have discovered that three of your teachers give out homework in different ways. How much homework will you have on your homework pile?

Record your information from your class. When you return to your home group, you will share your data with the other group members and you will receive the data from the other two classes.

Class A

Day	**Amount of work**	**First Differences**	**Second Difference**
1			
2			
3			
4			
5			

What type of relationship is this model? ____________________
Justify your answer.

Class B

Day	**Amount of work**	**First Differences**	**Second Difference**
1			
2			
3			
4			
5			

What type of relationship is this model? ________________________
Justify your answer.

5.1.2 Piles of Homework (continued)

Class C

Day	Amount of work	First Differences	Second Difference
1			
2			
3			
4			
5			

What type of relationship is this model? ___________________________

Is there a pattern to the first differences? Describe the pattern.

Is there a pattern to the second differences? Describe the pattern.

Revisiting The Paper – Folding Activity

Record your information from the paper-folding activity at the start of class. Complete the chart.

Fold	Number of layers	First Differences	Second Difference
0	1		
1			
2			
3			
4			
5			

Is there a pattern to the first differences and second differences? If yes, describe the pattern.

5.1.3 Exponential FRAME

Algebraic Model

Numerical Model

Visual/Spatial/Concrete

Description/Key words

Contextual

Graphical Model

Unit 5 : Day 2 : Investigating Exponential Growth

Grade 11 U/C

Minds On: 10

Action: 45

Consolidate:20

Total = 75 min

Description/Learning Goals

- Collect data through investigation from primary sources that can be modelled as exponential growth functions.
- Investigate secondary sources of data that can be modelled as exponential growth functions with and without the use of technology.
- Graph the data with and without technology.
- Identify exponential functions that arise from real-world applications involving growth.

Materials

- BLM 5.2.1
- BLM 5.2.2
- BLM 5.2.3
- BLM 5.2.4
- BLM 5.2.5 (Teacher)
- 4 guitars
- 4 measuring tapes
- 4 graphing calculators

Assessment Opportunities

🟩 **Minds On…**

Whole Class → Discussion

Describe the four centres (BLM 5.2.1, BLM 5.2.2, BLM 5.2.3, BLM 5.2.4) as necessary to prepare the students for the investigation. Connect the activity to the previous day's work on exponential functions.

🟥 **Action!**

Pairs → Investigation

Students will work in pairs and complete four different investigations related to exponential growth (BLM 5.2.1, BLM 5.2.2, BLM 5.2.3, BLM 5.2.4). Students will have approximately 10 minutes to work at each activity before moving to the next station. The students should be encouraged to collect all data and complete the graphs in the time that they have. They may need to take time to complete the questions related to the investigations at home. Ensure that all students record the data on their own observation sheets.

Mathematical Process Focus: **Representing** (Students will **represent** applications of Exponential Growth graphically, numerically, pictorially and concretely.)

🟦 **Consolidate Debrief**

Whole Class → Discussion

Direct the students to review the First Differences and constant ratios computed in BLM 5.2.1, BLM 5.2.2, BLM 5.2.3 and BLM5.2.4 and have students look for patterns.

Consider the following Guiding Questions:

a) Why does an exponential model fit the data you have examined?
b) How can you determine that the data is exponential from the table of values?
c) What was common in the shapes of the curves? (Sample response: They all increased over the domain, first slowly and then quickly.)

Add to the FRAME Model by including various representations of Exponential Growth.

Reflection

Home Activity or Further Classroom Consolidation

Students will reflect on the data they collected and consolidate their understanding by completing the questions provided on BLM 5.2.1 – 5.2.4 investigation sheets.

Students would benefit from a review on how to create a tree diagram and how to set up appropriate scales on axes for graphing.

Set up 4 stations for each activity. This gives 16 stations in total, which will accommodate 32 students.

Note: Suggested solutions for Teachers are found on BLM 5.2.5.

Literacy Strategy: Continue to add to the **Exponential FRAME Model.** The FRAME can be extended throughout the unit to develop the ideas further.

5.2.1 Investigating Exponential Growth: Pizza Toppings

INTRODUCTION

In this activity, you and your partner will investigate the number of different pizzas that can be created for a given number of available toppings.

INSTRUCTIONS

1. Fill in the chart below by drawing all the different pizzas that can be created by choosing all, some or none of the available toppings indicated.

Toppings Available	Different Pizza Drawings
none	
Cheese	
Cheese, Pepperoni	
Cheese, Pepperoni, Mushrooms	
Cheese, Pepperoni, Mushrooms, Bacon	

2. a) Use your information from question 1 to complete the table below

Number of Available Toppings	Number of Different Pizzas
0	
1	
2	
3	
4	

First Differences	Ratio

b) What do the First Differences tell us about this data? Why?

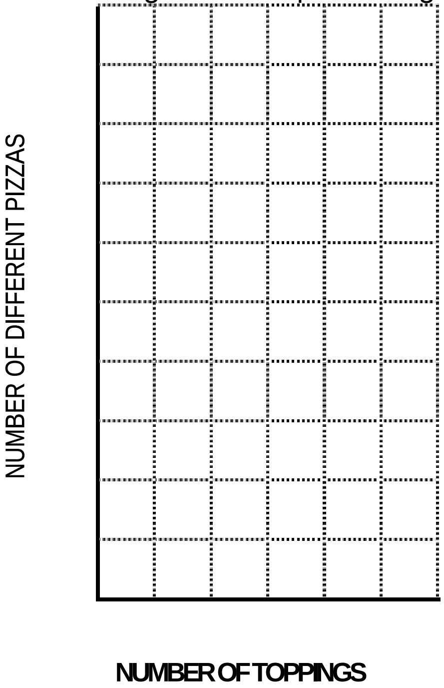
c) Is there a <u>constant</u> ratio between consecutive values in the column titled Number of Different Pizzas?

How does this value relate to the pattern in the First Differences?

c) Predict how many different pizzas could be created from 5 toppings. Justify your answer.

5.2.1 Investigating Exponential Growth: Pizza Toppings (continued)

3. Neatly sketch a graph of your results from question 2 on the grid below. Draw a smooth curve through the points. (Note: This is DISCRETE data; however, the smooth curve assists in seeing the general shape of the graph.)



4. Using the graph comment on the shape of the curve. Use words such as the following in your description: increasing, decreasing, quickly, slowly.

5. **Complete this statement**: As the number of toppings increases by 1, the number of different pizza combinations ___________________

6. Predict how many different pizzas can be created if there are nine available toppings. Clearly explain how you made your prediction.

7. If a restaurant owner would like to offer 200 different pizza combinations, what is the minimum number of available toppings she would need? Explain your reasoning.

8. a) Your local pizza parlour offers you the choice of 15 different toppings. If you were to eat a different pizza every day, how many years would it take for you to try every possible one? (Hint: There are 365 days in a year.)

   b) Does this answer surprise you? Why or why not?

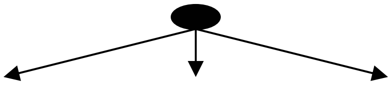
5.2.2 Investigating Exponential Growth: E-Mail Friendzy

INTRODUCTION

In this activity, you and your partner will use a Tree Diagram to simulate the effect of “telling three friends, who each tell three friends, who each tell three friends” and so on.

INSTRUCTIONS

1. A letter is e-mailed out to three friends. Each of these three recipients will then e-mail it to three new friends. Continue this pattern in order to complete the first four rounds in the Tree Diagram below. (Hint: Use tiny dots to represent each e-mail that is sent so that you will have enough space to draw out the entire Tree Diagram.)



Round 1

Round 2

Round 3

Round 4

2. a) Use the information from question 1 to complete the table.

Round	**Number of Letters e-mailed**
1	
2	
3	
4	
5	

First Differences	**Ratio**

b) Consider the Tree Diagram and the data in the first four rows of the table. Predict the Number of Letters e-mailed in Round 5. Justify your prediction.

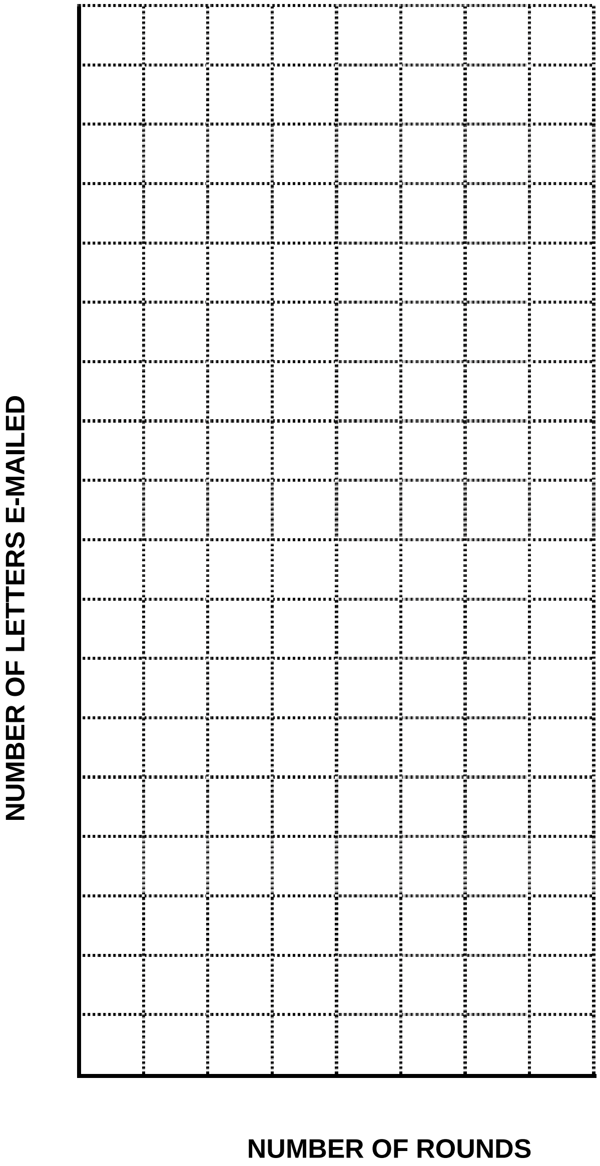
c) Comment on the patterns you see in the Number of Letters e-mailed column and the First Differences column.

d) Is there a constant ratio between consecutive values in the Number of Letters e-mailed column?

e) Comment on the value.

5.2.2 Investigating Exponential Growth: E-Mail Friendzy (continued)

3. Neatly sketch a graph of your results from question 2 on the grid below. Draw a smooth curve through the points. (Note: This is DISCRETE data; however, the smooth curve assists in seeing the general shape of the graph.)



4. **Complete this statement**: As the number of rounds increases by 1, the number of letters e-mailed ____________________

5. Predict how many letters will be e-mailed during the 9th round. Show how you determined this.

6. During which round will the number of letters e-mailed exceed 200 000 for the first time? Show how you determined this.

7. Pyramid schemes work similarly to this e-mail simulation in that individuals must find others willing to invest in (or purchase a product from) a “company”. Consider a pyramid scheme where people are asked to invest $1000 each and are required to find four more investors to do the same. How much **total** money will be invested in this “company” after three rounds? (Hint: Draw a tree diagram.) **NOTE: Pyramid schemes are illegal because they usually involve fraud.**

5.2.3 Investigating Exponential Growth: Guitar Frets

INTRODUCTION:

A fret is a ridge set across the fingerboard of a stringed instrument, such as a guitar. Frets divide the neck into segments at intervals related to a musical framework. In this activity, you and your partner will measure the distances between the frets and the bridge of a guitar. (See the diagram on the accompanying page.)

INSTRUCTIONS

1. Using a tape measure, accurately measure the distance for each indicated segment of the guitar as shown in the diagram on the accompanying page. All measurements must be taken in mm .

Measurement Number	Segment of Guitar	Distance (mm)
1	12th fret to bridge	
2	11th fret to bridge	
3	10th fret to bridge	
4	9th fret to bridge	
5	8th fret to bridge	
6	7th fret to bridge	
7	6th fret to bridge	
8	5th fret to bridge	
9	4th fret to bridge	
10	3rd fret to bridge	
11	2nd fret to bridge	
12	1st fret to bridge	

First Differences	Ratio

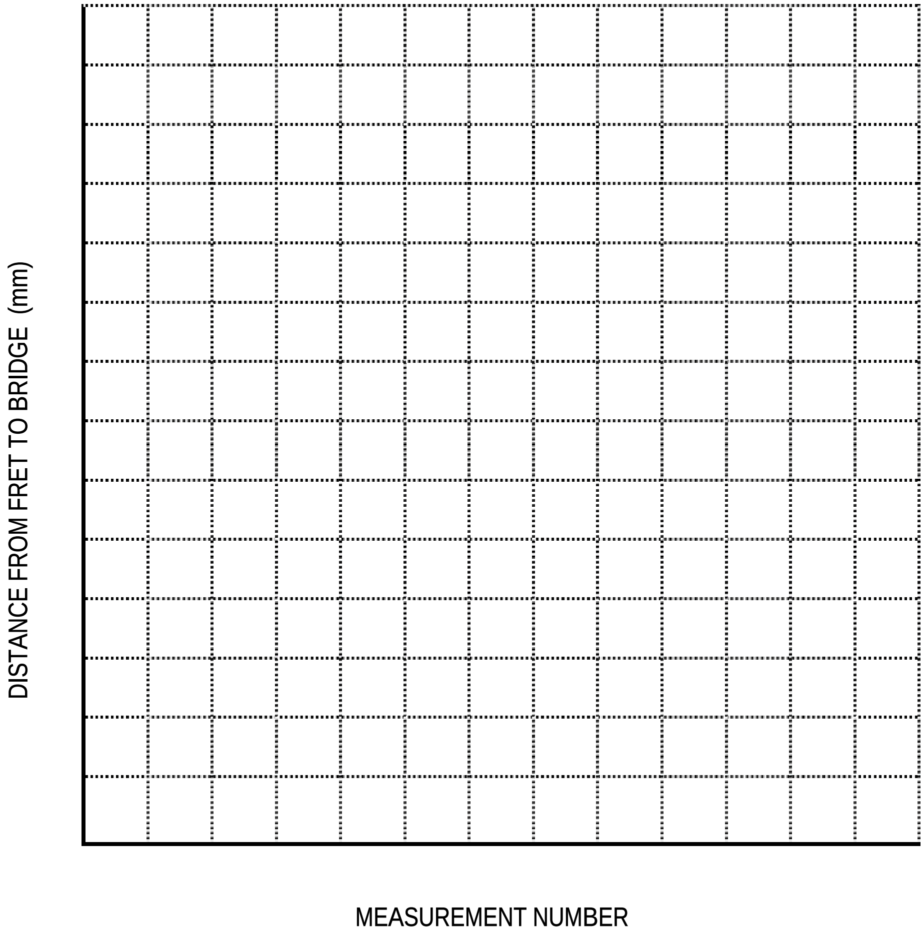
5.2.3 Investigating Exponential Growth: Guitar Frets (continued)

2. a) Is there a constant ratio between consecutive values in the column labelled Distance? If so, what is it? If not, explain why not.

   b) Predict how far the 13th fret is from the bridge and show how you made this prediction.

   c) Measure the distance from the 13th fret to the bridge and compare it to your prediction.

3. Neatly sketch a graph of your results from question 1 on the grid below. . Draw a smooth curve through the points.

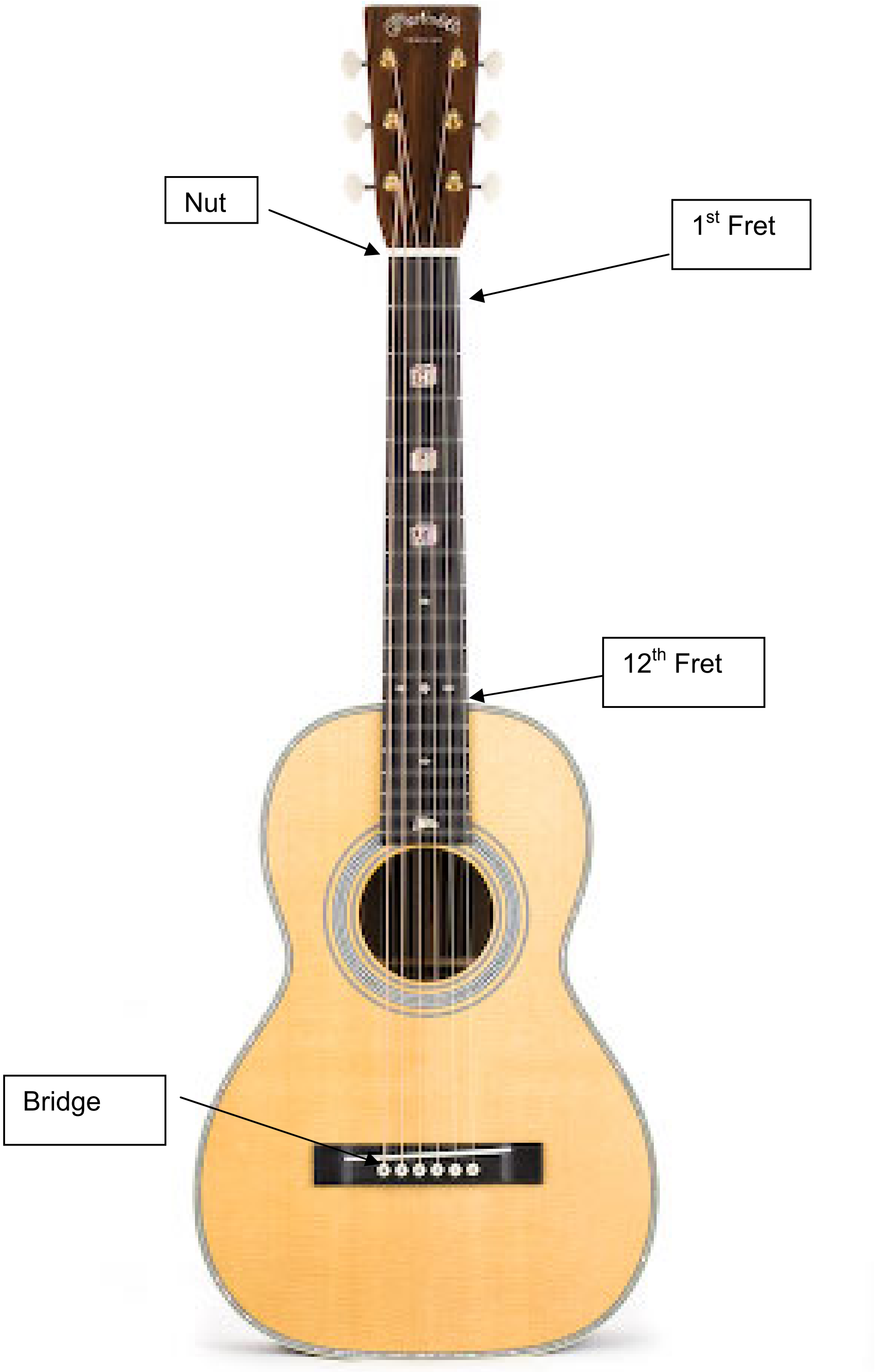


4. How would the graph change if the frets were equally spaced apart?

5.2.3 Investigating Exponential Growth: Guitar Frets (continued)

Diagram of Guitar Parts

Source: http://www.mandoweb.com/M_Claires%20Guitar.jpg)



5.2.4 Investigating Exponential Growth: Humpback Whale Population

(Source - http://www.nmfs.noaa.gov/pr/pdfs/sars/ao03humpbackwhalegulfofmaine.pdf)

INTRODUCTION If a population has a constant birth rate through time, and is never limited by food, disease or threat, it has what is known as exponential growth.

INSTRUCTIONS

1. The Humpback Whale is distributed worldwide in all ocean basins, though it is less common in Arctic waters. Barlow & Chapman (1997) have estimated a population growth rate of 6.5% per year for the well-studied Humpback Whale population in the Gulf of Maine. Through various methods, the Humpback Whale population in this region was estimated to be 652 whales in 1993

COMPLETE THE QUESTIONS IN THE BOX ON THE LEFT BEFORE FILLING OUT THE TABLE.

THINK:

a) What value do you multiply each population by if it is increasing by 6.5% per year?

b) Verify that your answer in (a) is correct by using it to calculate and confirm that the population for 1994 is 694 (rounded to the nearest whole number).

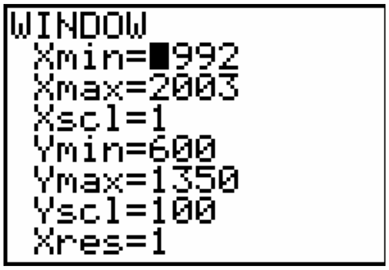
c) Complete the chart, by calculating the Population for each of the years in the table, using the growth rate of 6.5% per year. (Round answers to the nearest whole number.)

Year	Population	First Differences
1993	652	
1994	694	
1995		
1996		
1997		
1998		
1999		
2000		
2001		
2002		
2003		

5.2.4 Investigating Exponential Growth: Humpback Whale Population
(continued)

2. Enter the data from the chart above into the lists L_1 and L_2 of your graphing calculator.

3. Create a scatter plot of the data using your graphing calculator. Set your WINDOW as shown below and neatly sketch the plot as it appears on your calculator screen. Include labels and scales on your axes.



```
WINDOW
 Xmin=█992
 Xmax=2003
 Xscl=1
 Ymin=600
 Ymax=1350
 Yscl=100
 Xres=1
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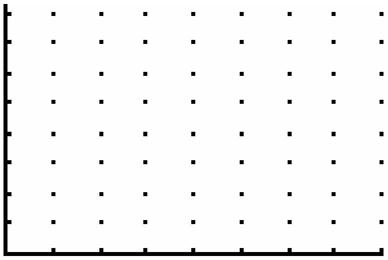


4. a) Does this data appear to be exponential from the graph? Why or why not?

   b) How would you convince someone that the data IS exponential?

5. Is there a relationship between the constant ratio of consecutive Population values and consecutive Finite Differences?

6. a) Predict the number of Humpback Whales that would be in the Gulf of Maine in the year 2006. Clearly explain how you made your prediction.

   b) Show how you could calculate the number of whales in the Gulf of Maine in 2006. Compare this result to your prediction.

5.2.5 Sample Solutions to Selected Growth Questions

BLM 5.2.1 Pizza Toppings

1. Students will draw the different pizza combinations in the chart. The first row with NO toppings available will have a single pizza (plain crust). The second row with CHEESE topping will have 2 pizzas (plain, cheese). The third row with Cheese and Pepperoni will have 4 pizzas (plain, cheese, pepperoni, cheese and pepperoni). The fourth row with Cheese, Pepperoni and Mushrooms will have 8 pizzas (plain, cheese, pepperoni, mushrooms, cheese and pepperoni, cheese and mushrooms, pepperoni and mushrooms, cheese and pepperoni and mushrooms). The fifth row will have 16 different pizzas.

2. b) Students will give varied responses such as: The First Differences tell us that the data is not linear (they aren't constant). The First Differences tell us that the data is exponential because there is a constant ratio between the consecutive first differences.

2. c) There is a constant ratio of 2 in the Number of Different Pizzas column. The First Differences also double.

2. d) 32. Sample response: I made the prediction by multiplying the value in row 4 by 2 (or I doubled the value in row 4).

3. Make a scatterplot. Have students join the points with a smooth curve in order to see the shape of the graph better. Discuss the fact that this is discrete data.

4. The curve increases slowly at first and then more quickly.

5. doubles

6. 512. Students will probably calculate the number of toppings by successively multiplying by 2. Some students may see the pattern as 2^9 but this can be discussed in more depth in the consolidation phase of the lesson.

7. Sample approach: Students would see that the answer to the previous question was 512, so half of this is 256. Therefore, 8 toppings are needed.

8. Calculate $\left(\frac{2^{15}}{365}\right) \approx 90$ years.

BLM 5.2.2 E-Mail Friendzy

1. The bottom row of the Tree Diagram (Round 4) will show 81 e-mails. Encourage students to use a small dot to represent each e-mail so that all 81 will fit in the space provided.

2. b) The number of e-mails is 243. Students may justify this by saying there is a constant ratio of 3. They may refer to the fact that there would be 3 e-mails sent from each of the 81 individuals in Round 4, meaning there would be 81x3=243 e-mails.

2. c) Students may see that both columns have a constant ratio of 3. They may comment on the fact that the First Differences are always twice as big as the values in the 2nd column (e.g., 6 is twice as big as 3, 18 is twice as big as 9, 54 is twice as big as 27 etc.)

2. d) e) There is a constant ratio of 3. This is the same as the number of e-mails sent by each person on each round.

5.2.5 Sample Solutions to Selected Growth Questions (continued)

3. Make a scatterplot. Have students join the points with a smooth curve in order to see the shape of the graph better. Discuss the fact that this is discrete data.

4. triples

5. 19683. Students will probably calculate the number of e-mails by successively multiplying by 3. Some students may see the pattern as 3^9 but this can be discussed in more depth in the consolidation phase of the lesson.

6. 12th round. Students will likely guess and test. This will depend on their level of understanding at this point. Some may test values of 3^n while others will use successive multiplication by 3.

7. The total value will be \$4000 + \$16 000 + \$64 000 = \$84 000. (Assuming the person at the top of the pyramid doesn't invest any money.)

BLM 5.2.3 Guitar Frets

1. Sample Chart showing data collected by a student. Note: Students need to be reminded to be as accurate as possible when taking the measurements.

Measurement	Distance (mm)
1 (12th fret)	321
2 (11th fret)	340
3 (10th fret)	361
4 (9th fret)	382
5 (8th fret)	405
6 (7th fret)	430
7 (6th fret)	455
8 (5th fret)	482
9 (4th fret)	511
10 (3rd fret)	542
11 (2nd fret)	574
12 (1st fret)	609

Note: The constant ratio is approximately 1.06. The variation is due to measurement error.

2. a) The constant ratio is close to 1.06

2. b) To find the distance from the 13th fret, we go the other way in the table, meaning the constant ratio is about 0.94. Therefore, the 13th fret should be 321 x 0.94 = 301 mm from the bridge.

3. Make a scatterplot. Have students join the points with a smooth curve in order to see the shape of the graph better. Discuss the fact that this is discrete data.

4. The graph would be linear if the frets were equally spaced apart.

BLM 5.2.4 Humpback Whales

1. a) 1.065
   b) Students will show 652 x 1.065 is approximately 694.
   c) The bottom row will show there will be about 1224 whales by the year 2003.

4. a) b)The students may observe that the graph is increasing slowly at first, with a bit of an increase towards the end. Note: Some students may feel this graph looks more linear. This would lead to a good discussion as to what other information we have (besides the graph) that would help us to determine if this is linear or not. (e.g., First Differences, constant ratio)

5. Yes. They are both 1.065.

6. Students will probably use successive multiplication by 1.065. The number will be close to 1480 depending on the rounding used by the students.

Unit 5 : Day 3 : Investigating Exponential Decay

Grade 11U/C

Minds On:	10
Action:	45
Consolidate:	20
Total	= 75 min

Description/Learning Goals

- Collect data through investigation from primary sources that can be modelled as exponential decay functions.
- Investigate secondary sources of data that can be modelled as exponential decay functions with and without the use of technology.
- Graph the data with and without technology.
- Identify exponential functions that arise from real-world applications involving decay.

Materials

- BLM 5.3.1
- BLM 5.3.2
- BLM 5.3.3
- BLM 5.3.4
- BLM 5.3.5 (Teacher)
- 400 dice
- 4 containers for shaking dice
- 4 box lids

Assessment Opportunities

🟩 Minds On…

Whole Class → Discussion

Describe the four activities (BLM 5.3.1, BLM 5.3.2, BLM 5.3.3, BLM 5.3.4) in which the students will be engaging and connect the activity to the previous day's work on exponential functions.

Set up 4 stations for each activity. This gives 16 stations in total, which will accommodate 32 students.

🟥 Action!

Pairs → Investigation

Students will work in pairs and complete four different investigations related to exponential growth (BLM 5.3.1, BLM 5.3.2, BLM 5.3.3, BLM 5.3.4). Students will have approximately 10 minutes to work at each activity before moving to the next station. The students should be encouraged to collect all data and complete the graphs in the time that they have. They may need to take time to complete the questions related to the investigations at home. Ensure that all students record the data on their own observation sheets.

Mathematical Process Focus: **Representing** (Students will **represent** applications of Exponential Growth graphically, numerically, pictorially and concretely)

Learning Skill/Team Work/Observation/Rating Scale: By observation, assess and record each student's level of teamwork using a four point scale (N,S,G,E).

Note: Suggested solutions for Teachers are found on BLM 5.3.5.

Suggestion: For the Radioactive Atoms (BLM 5.3.2) use a Pringles-type chip can as the container. Have students roll the dice into a box lid to contain the dice.

🟦 Consolidate Debrief

Whole Class → Discussion

The teacher should direct the students to review the First Differences and constant ratios computed in BLM 5.3.1, BLM 5.3.2, BLM 5.3.3 and BLM 5.3.4 and have students look for patterns.

Consider the following Guiding Questions:

a) Why does an exponential model fit the data you have examined?

b) How can you determine that the data is exponential, from the table of values?

c) What was common in the shapes of the curves? (Sample response: They all decreased over the domain, first quickly and then slowly.)

Add to the FRAME Model by including various representations of Exponential Decay.

Literacy Strategy: Continue to add to the **Exponential FRAME Model.**

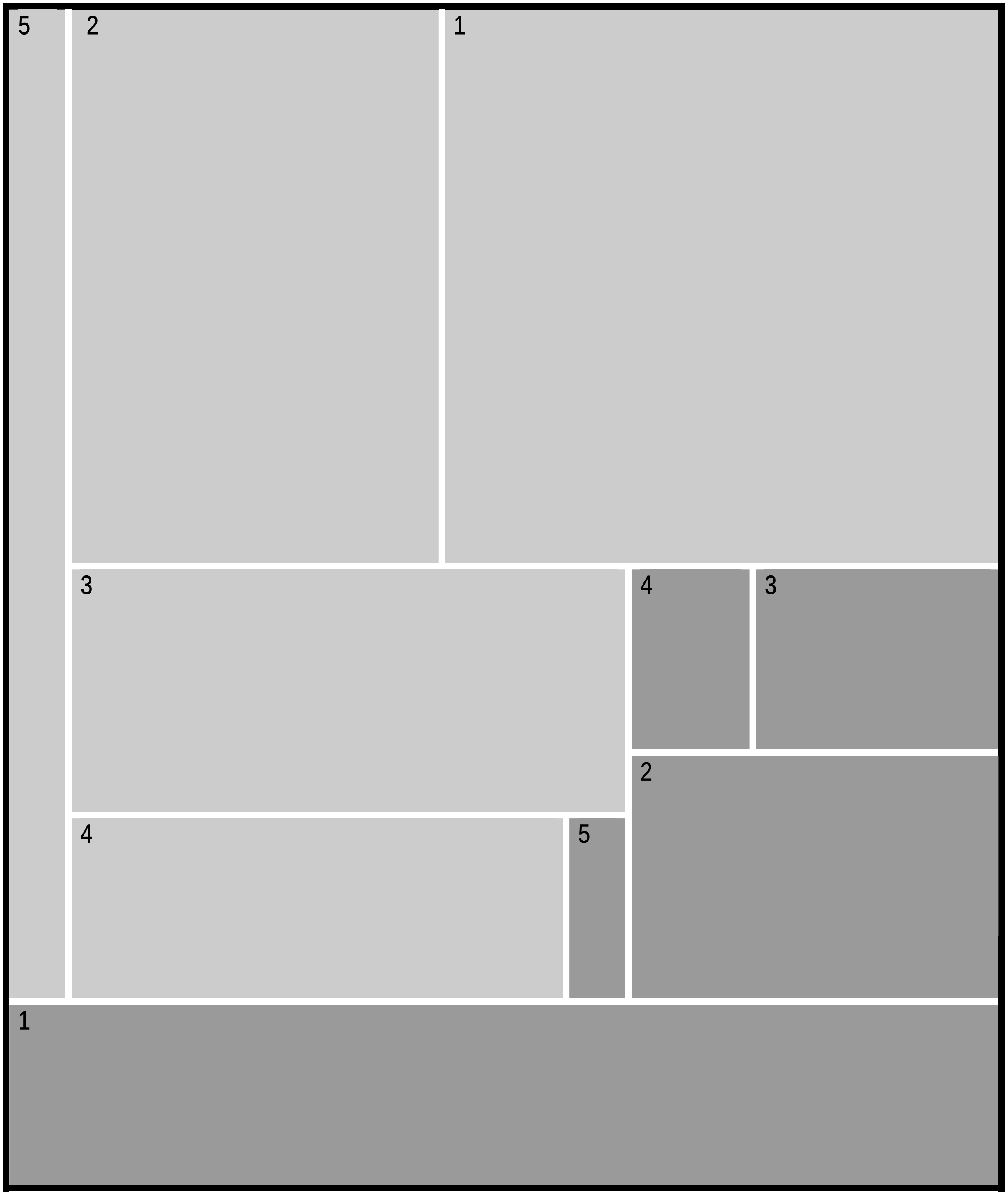
Reflection

Home Activity or Further Classroom Consolidation

Students will reflect on the data they collected and consolidate their understanding by completing the questions provided on BLM 5.3.1-5.3.4 investigation sheets.

5.3.1 Investigating Exponential Decay: Tiling a Floor

INTRODUCTION: A lavish hotel is undergoing a renovation and has hired an interior design team to finish the very large rectangular floor in its foyer. It has been decided to use light and dark grey marble slabs to cover the 16 m by 19 m surface. (See diagram below.)



5.3.1 Investigating Exponential Decay: Tiling a Floor (continued)

INSTRUCTIONS

1. Using the scale 1 cm represents 1 m and a ruler, determine the actual length, width and area for each indicated dark grey slab and record your results in the table below.

Dark Grey Slabs			
Slab Number	**Length (m)**	**Width (m)**	**Area (m^2)**
1			
2			
3			
4			
5			

First Differences	Ratio

2. Repeat step 1 for each indicated light grey slab and record your results in the table below.

Light Grey Slabs			
Slab Number	**Length (m)**	**Width (m)**	**Area (m^2)**
1			
2			
3			
4			
5			

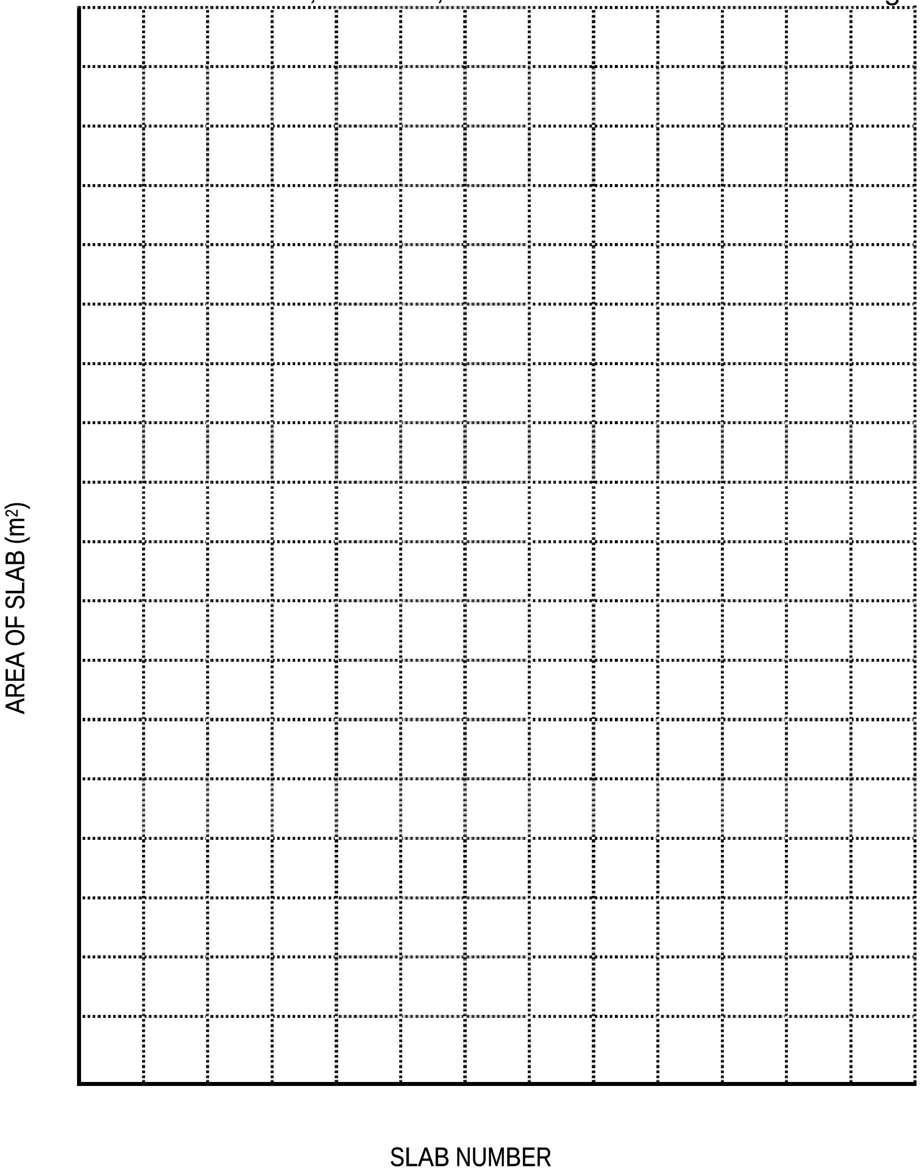
First Differences	Ratio

3. Contrast these First Differences with those of exponential growth data?

4. Calculate the ratios between consecutive Areas in each table. What do these values tell us about the data?

5.3.1 Investigating Exponential Decay: Tiling a Floor (continued)

5. Neatly sketch 2 graphs on the same grid below using your data from the tables. (Note: This is DISCRETE data; however, the smooth curve assists in seeing the general shape of the graph.)



6. Refer back to your table and calculate the **total** area covered by each type of marble.

7. The light grey marble costs \$ 700/m ^{2} and the dark grey marble costs \$ 950 /m ^{2}. Determine the amount of money saved by using light grey for the whole foyer area instead of covering the area with the combination of light and dark grey marble slabs as shown in the diagram.

5.3.2 Investigating Exponential Decay: Radioactive Atoms

INTRODUCTION

All matter is made up of atoms. Some kinds of atoms have too much energy and are unstable. These atoms are called radioactive. It is not possible to predict exactly when a radioactive atom will release its extra energy and form a different stable atom. This process is called radioactive decay. In this experiment you will model this process. (Source: Ministry of Education Regional Training Sessions, Grade 11 Mathematics, Spring 2006)

INSTRUCTIONS

1. Fill your container with 100 dice. Each die represents one atom.

2. a) Pour the dice out of the container onto a tray. The atoms with the number one facing up have decayed. Remove them and **record the number of atoms remaining** in the table below. Repeat the process until there are five or fewer atoms remaining. Note that each 'pour' represents one time period (we will call each time period a day). Extend the table if needed.

Number of Days	Number of Atoms Remaining	First Differences
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

b) How do these First Differences contrast with the First Differences of exponential growth data?

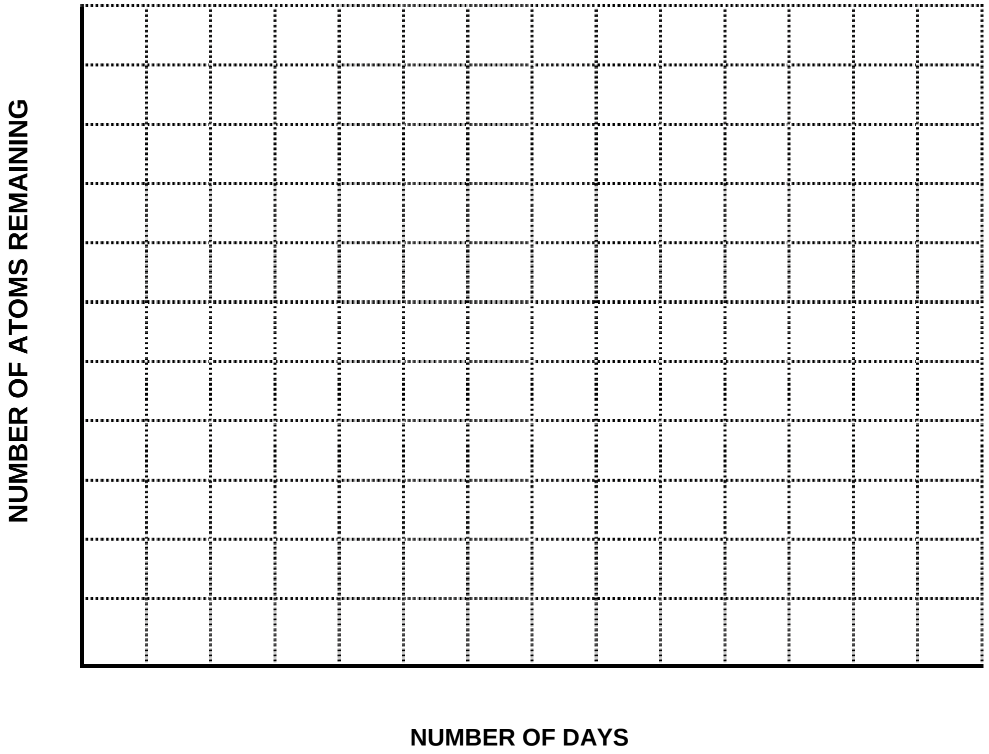
c) Calculate the ratio between the first two values in the column titled Number of Atoms Remaining. What does this ratio mean in the context of this problem?

d) Calculate the ratio between the 2nd and 3rd values in the column titled Number of Atoms Remaining. Compare it to the ratio from (c). Account for any differences.

5.3.2 Investigating Exponential Decay: Radioactive Atoms

(continued)

3. Neatly sketch a graph of your data. Draw a smooth curve through the points.



4. The time it takes for half of the original number of atoms to decay is called the half-life. Use your graph above to determine the half-life of your substance in days.

5. When one half-life has passed there will be 50 atoms of the original radioactive substance remaining. Use your graph to determine how long it will take for these 50 atoms to reduce to 25 atoms.

6. Use your graph to determine how long it will take for the 25 atoms to reduce to about 12 (half of 25).

7. What can you conclude about the half-life of your atoms.

5.3.3 Investigating Exponential Decay: Car Depreciation

INTRODUCTION: Depreciation is the decline in a car's value over the course of its useful life. It's something new-car buyers dread. Most modern domestic vehicles typically depreciate at a rate of 15%-20% per year depending on the model of the car.

INSTRUCTIONS

1. A 2007 Ford Mustang GT convertible is valued at $32 000 and depreciates on average at 20% per year.

   Since the car is depreciating at 20% per year, the remaining value at the end of the first year is ____% of the original value.

   Therefore, to find the depreciated value, multiply the previous year's value by _______.

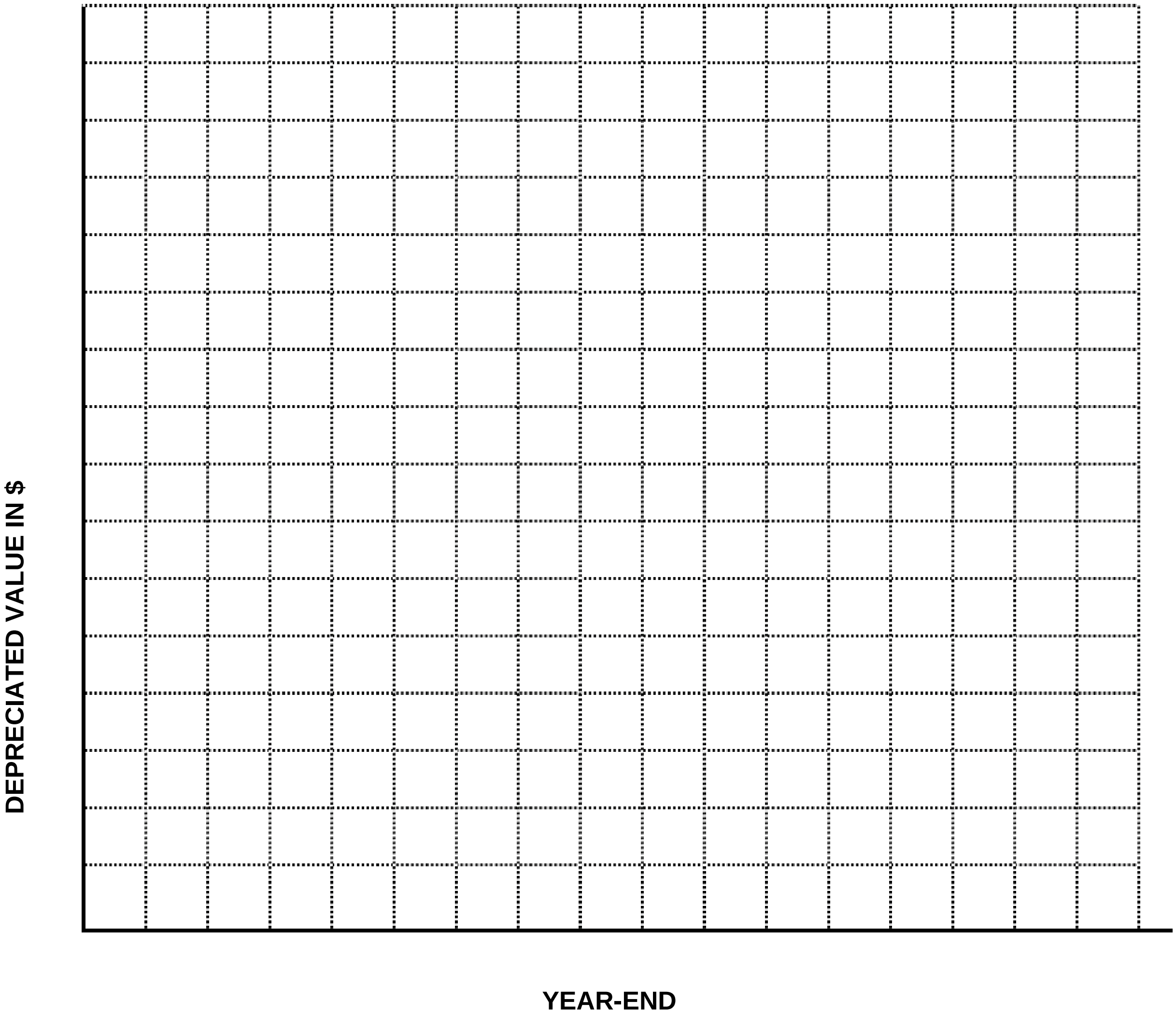
2. Complete the following table to calculate the value of the car at the end of each of the first five years of ownership.

Year-end	Depreciated Value in $	Show Calculation in this column
0	32 000	▨
1		
2		
3		
4		
5		

3. What would you expect the constant ratio to be for this example. Justify your answer.

5.3.3 Investigating Exponential Decay: Car Depreciation (continued)

4. Neatly sketch a graph of your data. Draw a smooth curve through the points.



5. If the depreciation was 15% per year, how would the constant ratio change?

5.3.4 Investigating Exponential Decay: Computers in Schools

INTRODUCTION: Computers have become an integral part of today's schools. However, this has not always been the case. The table below shows the average number of students per computer in public schools (elementary and secondary) since 1983. (Source of data: **Precalculus: A Graphing Approach**, Holt, Rinehart and Winston 2002, p.398)

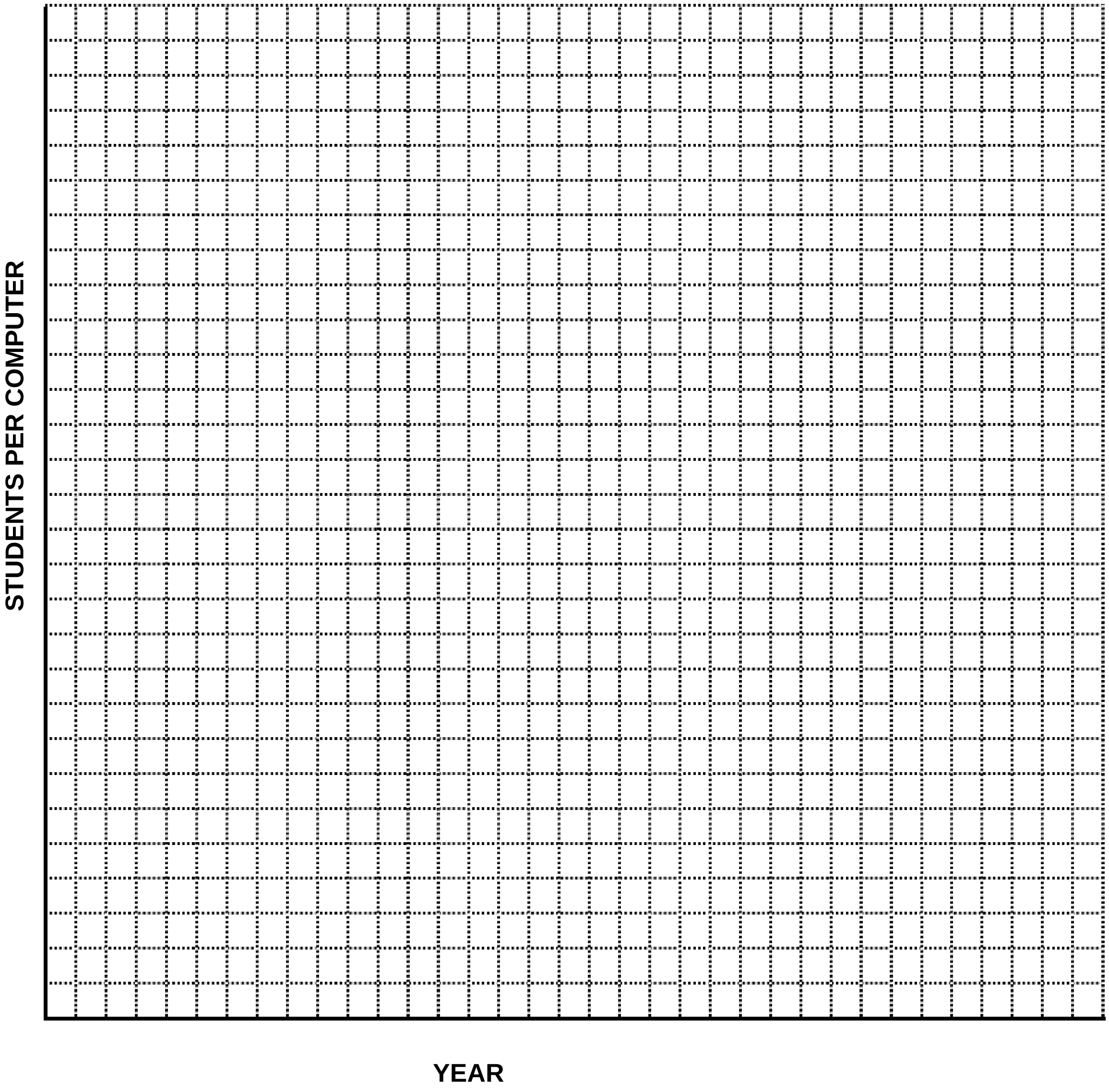
Year	Population	First Differences	Ratio
1983	125		
1984	75		
1985	50		
1986	37		
1987	32		
1988	25		
1989	22		
1990	20		
1991	18		
1992	16		
1993	14		
1994	10.5		
1995	10		

INSTRUCTIONS

1. Examine the table above. How is this data different from the exponential growth data?

5.3.4 Investigating Exponential Decay: Computers in Schools (continued)

2. Neatly sketch a graph of the data from the table on the previous page. When choosing your scale for the horizontal axis, consider question 4 below. After you have plotted the points, draw a smooth curve through them.



3. Using the graph, comment on the shape of the curve. Use words such as the following in your description: increasing, decreasing, quickly, slowly.

4. Use your graph to predict the number of students per computer in the year 2006.

5. Is the answer from question #4 surprising? Why or why not?

5.3.5 Sample Solutions to Selected Decay Questions

BLM 5.3.1 Tiling a Floor

3. These First Differences are negative, whereas the First Differences for exponential growth data were positive.
4. The constant ratio of $\frac{1}{2}$ in the first table tells us that the areas of consecutive dark grey slabs are being cut in half. The constant ratio of $\frac{2}{3}$ in the second table tells us that each consecutive light grey slab has an area that is $\frac{2}{3}$ of the previous slab.
6. The total dark grey area is 93 square metres and the total light grey area is 211 metres.
7. The dark grey cost is 93 x $950 = $88350. The light grey cost is 211 x $700 = $147 700. Therefore the total is: $88350 + $147 700 = $236 050. If the whole foyer is completed using the light grey, then the total area is 93 + 211 = 304 square metres at a cost of $700 per metre is: $212 800. The cost savings is: $236 050 - $212 800 = $23 250.

BLM 5.3.2 Radioactive Atoms

2. a) Values in the table will vary depending on the data collected.

2. b) The First Differences are negative, whereas the First Differences for exponential growth data were positive.

2. c) The ratio will vary depending on the data. It should be around 83% (5/6 of the atoms would theoretically remain each time.) The rate represents the percentage of atoms remaining after each day.

2. d) There will be differences due to experimental factors.

3. Make a scatterplot. Have students join the points with a smooth curve. Discuss that this is continuous data as the atoms would theoretically be decaying all the time (not just once a day.)

4. The answers will vary depending on the data. The half life should be about 4-5 days.

5. Answers to 5 ,6 and 7 will also vary depending on the data.

BLM 5.3.3 Car Depreciation

1. The remaining value at the end of the first year is 80% of the original value. Therefore, multiply the previous year's value by 0.80.

2. The value at the end of year 5 will be $10485.76.

3. The constant ratio will be 0.80. This is the value we used to create the data.

4. Draw a smooth curve through the points. Discuss that this is continuous data.

5. The constant ratio would be 0.85.

5.3.5 Sample Solutions to Selected Decay Questions (continued)

BLM 5.3.4 Computers in Schools

1. The number of students per computer is DECREASING instead of INCREASING.
2. Teachers may wish to discuss with students why this data is continuous.
3. The curve is decreasing quickly at first and then slowly
4. The answer will depend on the accuracy of the graph. Answers may range from 1 to 3. The answer to #5 will vary depending on the student's understanding of exponential decay and also on the connection they make to the real world with this problem (e.g., understanding that the number of computers in schools has increased dramatically over the last decade).

Unit 5 : Day 4 : Investigating the Graphs of Exponential Functions – Part 1

Grade 11U/C

Minds On: 10

Action: 50

Consolidate:15

Total=75 min

Description/Learning Goals

- Graph exponential functions in the form $y = ab^x$ where $b > 0$ and $a = 1$.
- Identify the key features (y-intercept, increasing or decreasing, domain and range, horizontal asymptote).
- Use the table of values to show the pattern of the first differences and to calculate the constant ratio.

Materials

- Graphing calculators
- BLM 5.4.1
- post-it notes

Assessment Opportunities

🟩 **Minds On…**

Whole Class → Discussion

Activate prior knowledge related to exponential functions by collectively creating a “class” KWL chart (Think Literacy, Mathematics 10-12, p.54).
Create a KWL template on the board or on chart paper.
Give each student 4 post-it notes. Students individually write ideas they already KNOW related to exponential functions on 3 of the notes. On the fourth they record something they are WONDERING about.
Students place their notes under the K and W on the class chart.
The teacher can group the common ideas together and summarize with the class.

🟥 **Action!**

Individual → Investigation Using Technology

Students will complete BLM 5.4.1 to identify the shapes and properties of exponential functions. Students will complete tables of values and look for patterns in the data by calculating the first differences and the constant ratio.

Mathematical Process Focus: **Connecting** (Students will make the **connection** between the value of the base of the exponential function and the shape of the graph. Students will make the **connection** between the constant ratio and the base of the exponential function. Students will make **connections** between different representations: algebraic, graphical, numeric)

It is important to distinguish between finding a difference and finding a ratio. Students will see/review that the ratio for exponential functions can be calculated from the y coordinates or finite differences.

At the end of the next lesson the students will be given 2 post-it notes to record the things they have LEARNED.

🟦 **Consolidate Debrief**

Whole Class → Discussion

Students will share their responses to the consolidation questions at the end of the activity.
Students will explain why the relationships are functions.
Summarize the similarities in the graphs (domain, range, y-intercept, regions of increase and decrease).
Identify the horizontal asymptote.
Students will examine the patterns in the y-coordinates in the table of values and in particular the use of the ratio in predicting the y-coordinate when x is a negative integer.

Add to the **FRAME** model by including examples of equations.

This is the first time a horizontal asymptote has been identified.

Refer to this pattern in the future lesson on evaluating numerical expressions containing integer exponents.

Literacy Strategy: Continue to add to the **Exponential FRAME Model.**

Application
Concept
Practice

Home Activity or Further Classroom Consolidation

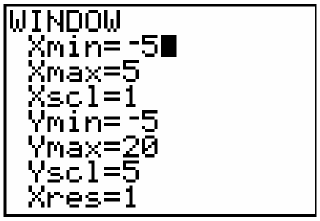
a) Without graphing or making a table of values predict the properties (domain, range, y-intercept, regions of increase and decrease, growth or decay, horizontal asymptote) of the graphs of $y = 5^x$ $y = \left(\frac{1}{5}\right)^x$.

b) To verify your predictions, make tables of values and graph on the same grid. What other observations can you make about the graphs?

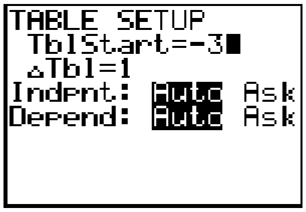
5.4.1 The Graphs of Exponential Functions – Part 1

Step 1:

- Set your graphing calculator to the following window and the TBLSET menu to the following settings.



```
WINDOW
 Xmin=-5
 Xmax=5
 Xscl=1
 Ymin=-5
 Ymax=20
 Yscl=5
 Xres=1
```



```
TABLE SETUP
 TblStart=-3
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

Step 2:

- Each of the equations is in the form: $y = ab^x$.
- Graph each equation using the calculator.
- Calculate the values in the chart for zero and positive x-values. Use the TABLE feature of the calculator to verify your work. Copy the y-value for negative x-values from the TABLE.
- Label and put the scale on each axis and carefully graph the relationship.
- Calculate the first differences, Δy, and the constant ratio.

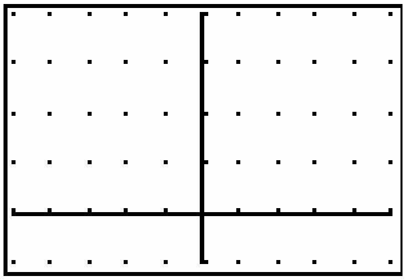
1. $y = 2^x$

y-intercept is
a =
b =

increasing, decreasing or neither (circle one)

Domain =

Range =



x	y
-1	
0	
1	
2	
3	
4	

∊y	Ratio

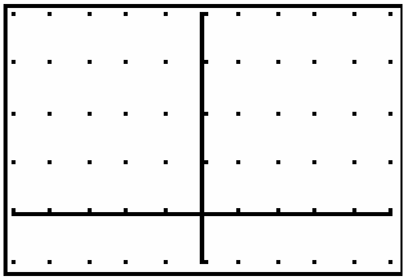
2. $y = 3^x$

y-intercept is
a =
b =

increasing, decreasing or neither (circle one)

Domain =

Range =



x	y
-1	
0	
1	
2	
3	
4	

∊y	Ratio

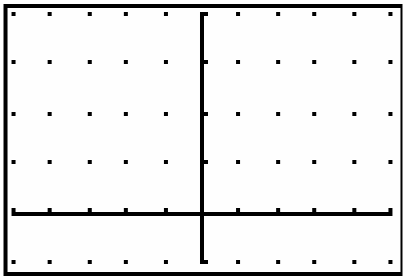
3. $y = 4^x$

y-intercept is
a =
b =

increasing, decreasing or neither (circle one)

Domain =

Range =



x	y
-1	
0	
1	
2	
3	
4	

∊y	Ratio

5.4.1 The Graphs of Exponential Functions – Part 1 (continued)

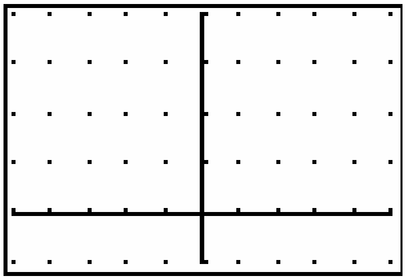
4. $y = 1^x$

y-intercept is
a =
b =

increasing, decreasing or neither (circle one)

Domain =

Range =



x	y
-1	
0	
1	
2	
3	
4	

∈y	Ratio

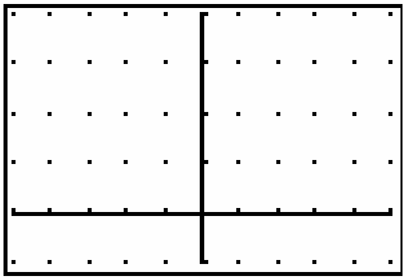
5. $y = \left(\frac{1}{2}\right)^x$

y-intercept is
a =
b =

increasing, decreasing or neither (circle one)

Domain =

Range =



x	y
-4	
-3	
-2	
-1	
0	
1	

∈y	Ratio

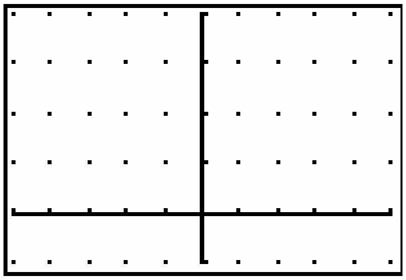
6. $y = \left(\frac{1}{3}\right)^x$

y-intercept is
a =
b =

increasing, decreasing or neither (circle one)

Domain =

Range =



x	y
-3	
-2	
-1	
0	
1	
2	

∈y	Ratio

Step 3: Consolidation

1. What ordered pair do these graph have in common?____________

2. Make a chart listing the equations that have graphs that are increasing, decreasing and neither increasing nor decreasing.

Increasing	Decreasing	Neither

What characteristic does an exponential equation have if its graph
a) Increases? b) Decreases?

Explain why there is a graph that shows neither exponential growth nor decay.

3. Describe how the value of the base, *b*, of an exponential function determines the shape of the graph.

Unit 5 : Day 5 : Investigating the Graphs of Exponential Functions – Part 2

Grade 11 U/C

Minds On: 10

Action: 40

Consolidate:25

Total=75 min

Description/Learning Goals

- Graph exponential functions in the form $y = ab^x$,where $b > 0$ and $a > 1$.
- Reinforce the key features (y-intercept, increasing or decreasing, domain, range and horizontal asymptote).
- Use the table of values to show the pattern of the first differences and to calculate the constant ratio.

Materials

- Graphing calculators
- BLM 5.5.1
- BLM 5.5.2 (Teacher)
- post-it notes

Assessment Opportunities

🟩 **Minds On…**

Whole Group → Discussion

Using a Venn Diagram, compare the properties of the functions that were investigated as a home activity. Group the similarities and the differences. Note the relationship between the graphs.

Literacy Strategy: See BLM 5.5.2 for Sample Venn Diagram.

🟥 **Action!**

Individual → Investigation Using Technology

Students will complete BLM 5.5.1 to identify the shapes and properties of exponential functions.

Curriculum Expectation/Demonstration/Checklist: Assess students' understanding of exponential functions from their graphs.

Mathematical Process Focus: Connecting (Students will make the **connection** between the value of a and the effect on the y-intercept that occurs for the function $y = ab^x$ compared with the function $y = b^x$.)

Mathematical Process Focus: Reasoning and Proving (See Step 3 Consolidation on BLM 5.5.1.)

🟦 **Consolidate Debrief**

Whole Class → Discussion

Students will share their responses to the consolidation questions at the end of the activity.

Revisit the data collection activities and justify responses to the following Guiding Questions:

- Which exponential functions have b>1?
- Which exponential functions have 0<b<1?
- Which exponential functions are in the form $y = ab^x$, $y = b^x$?
- In what situations is it difficult to decide from a graph whether the relationship is linear or exponential?

Distribute 2 post-it notes to each student and complete the LEARNED section on the class KWL chart.

Add to the FRAME Model by including examples of equations.

Literacy Strategy: Complete the class KWL chart.

Literacy Strategy: Continue to add to the **Exponential FRAME Model.**

Exploration

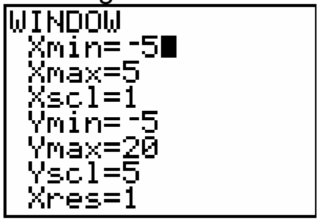
Home Activity or Further Classroom Consolidation

- Make tables of values for each of the following functions $y = 2^x$, $y = 3(2)^x$ and $y = 6^x$ .Graph on the same on the same axis.
- Make connections between the graphs.

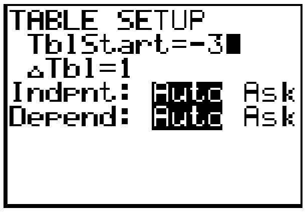
5.5.1 The Graphs of Exponential Functions – Part 2

Step 1:

- Set your graphing calculator to the following window and the TBLSET menu to the following settings.



```
WINDOW
 Xmin=-5
 Xmax=5
 Xscl=1
 Ymin=-5
 Ymax=20
 Yscl=5
 Xres=1
```



```
TABLE SETUP
 TblStart=-3
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

Step 2:

- Each of the equations is in the form: $y = ab^x$.
- Graph each equation using the calculator.
- Calculate the values in the chart for zero and positive x-values. Use the TABLE feature of the calculator to verify your work. Copy the y-value for negative x- values from the TABLE.
- Label and put the scale on each axis and carefully graph the relationship.
- Calculate the first differences, Δy, and the constant ratio.

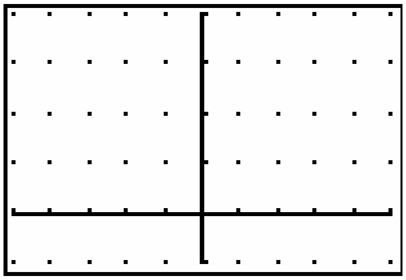
1. $y = 3(2^x)$

y-intercept is
a =
b =

increasing, decreasing or neither (circle one)

Domain =

Range =



x	y
-1	
0	
1	
2	
3	
4	

∊y	Ratio

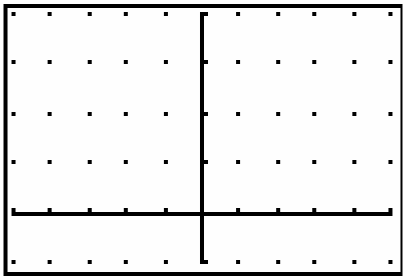
2. $y = 2(3^x)$

y-intercept is
a =
b =

increasing, decreasing or neither (circle one)

Domain =

Range =



x	y
-1	
0	
1	
2	
3	
4	

∊y	Ratio

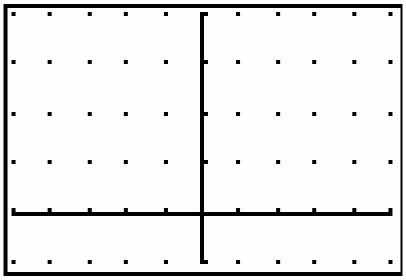
3. $y = 3(4^x)$

y-intercept is
a =
b =

increasing, decreasing or neither (circle one)

Domain =

Range =



x	y
-1	
0	
1	
2	
3	
4	

∊y	Ratio

5.5.1 The Graphs of Exponential Functions – Part 2 (continued)

4. $y = 5\left(\frac{1}{2}\right)^x$

y-intercept is

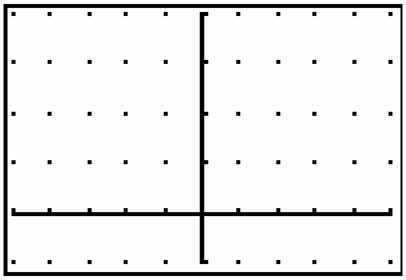
a =

b =

increasing, decreasing or neither (circle one)

Domain =

Range =



x	y
-4	
-3	
-2	
-1	
0	
1	

∊y	Ratio

5. $y = 4\left(\frac{1}{3}\right)^x$

y-intercept is

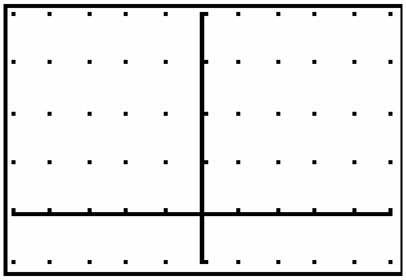
a =

b =

increasing, decreasing or neither (circle one)

Domain =

Range =



x	y
-3	
-2	
-1	
0	
1	
2	

∊y	Ratio

Step 3: Consolidation

1. Explain the significance of the y-intercept in each of the graphs.

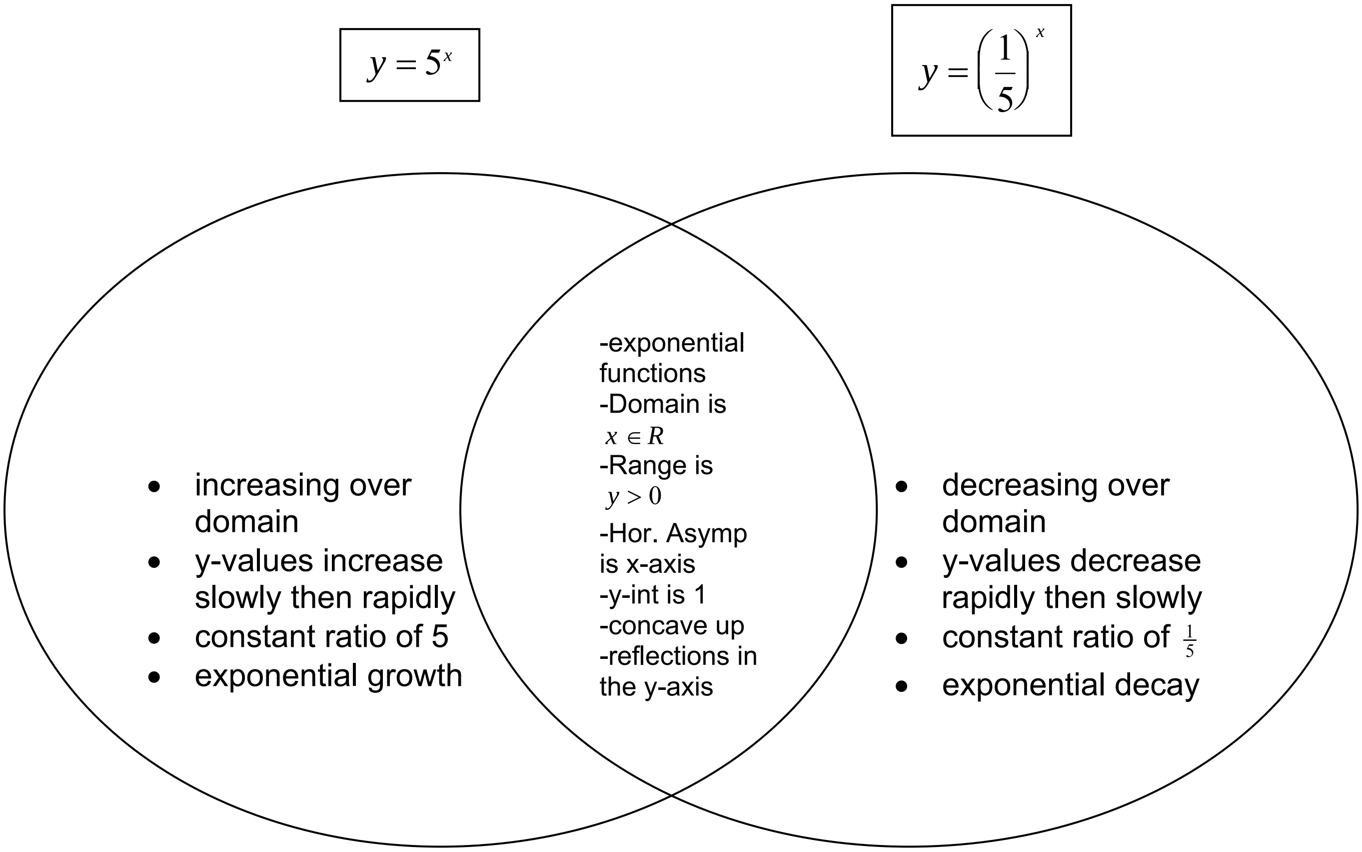
2. Consider the following data.

x	y
0	4
1	20
2	100
3	500
4	2500
5	12500

a) Show clearly the steps you would follow to determine the equation of the function that would model this data.

b) Prove that the function you have found does generate the data in the table.

5.5.2 Sample Venn Diagram (Teacher Copy)



$y = 5^x$

- increasing over domain
- y-values increase slowly then rapidly
- constant ratio of 5
- exponential growth

-exponential functions
-Domain is $x \in R$
-Range is $y > 0$
-Hor. Asymp is x-axis
-y-int is 1
-concave up
-reflections in the y-axis

$y = \left(\frac{1}{5}\right)^x$

- decreasing over domain
- y-values decrease rapidly then slowly
- constant ratio of $\frac{1}{5}$
- exponential decay

Unit 5: Day 6 : Domain and Range in Real-World Applications

Grade 11U/C

Minds On: 10

Action: 40

Consolidate:25

Total=75 min

Description/Learning Goals

- Identify exponential functions that arise from real world applications involving growth and decay given tables of values, graphs and/or equations.
- Determine reasonable restrictions on the domain and range of real world applications modelled by exponential functions including the applications investigated on Days 2 and 3.
- Distinguish exponential functions from linear and quadratic functions.

Materials

- BLM 5.6.1
- BLM 5.6.2

Assessment Opportunities

Minds On…

Whole Class → Discussion

Refer students to BLM 5.2.1 (Pizza Toppings) and ask "In the context of this application, are there any restrictions on the domain (number of toppings) and range (number of different pizza combinations)?" Discuss with students the effect of discrete or continuous data on the domain and range. Have students record the domain and range, with reasons, on BLM 5.6.1.

Action!

Pairs → Connecting

Students complete worksheet BLM 5.6.1 in pairs, revisiting the applications investigated on Days 2 and 3 and determining a reasonable domain and range for each application.

Small Groups → Presentations

Divide students into seven groups using Numbered Heads. Assign each group one of the applications from BLM 5.6.1 (not Pizza). Give students 5 minutes to discuss their domain and range values for the application, with reasons, and agree on a common set of restrictions to present to the class. Each group will have 2 minutes to present and respond to feedback from the class.

For more information of the group organizer, Numbered Heads, refer to Beyond Monet, Bennett, p. 106.

Mathematical Process Focus: Reflecting: (Students will **reflect** on the domain and range for applications that can be modelled by exponential functions, and explain any restrictions.)

Curriculum Expectation/Demonstration/Observation: Assess students' ability to determine reasonable restrictions on the domain and range of real world applications as they present.

Pairs → Peer Coaching

Students work in pairs to complete Part A of BLM 5.6.2. A coaches B and B coaches A

Consolidate Debrief

Whole Class → Discussion

Students share their answers from Part A of BLM 5.6.2.

Pose the following Guiding Questions:

a) Why do we need to restrict the domain and range of real-world applications modelled by exponential functions?

b) How can reasonable restrictions be determined on the domain and range of real-world applications?

c) How do restrictions placed on the domain relate to restrictions on the range?

Add to the FRAME Model by including additional information on domain and range.

Literacy Strategy: Continue to add to the **Exponential FRAME Model.**

Application
Concept
Practice
Reflection

Home Activity or Further Classroom Consolidation

Complete Part B of worksheet BLM 5.6.2.

5.6.1 Restricting Domain and Range in Real-World Applications

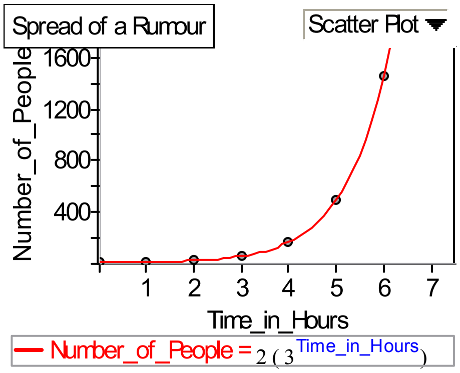
Application	Restrictions on Domain (with reasons)	Restrictions on Range (with reasons)
Pizza Toppings		
E-mail Friendzy		
Guitar Frets		
Humpback Whales Population		
Tiling a Floor		
Radioactive Atoms		
Car Depreciation		
Computers in Schools		

5.6.2 Are you the Master of Your Domain and Range?

Part A: Peer Coaching

A coaches B

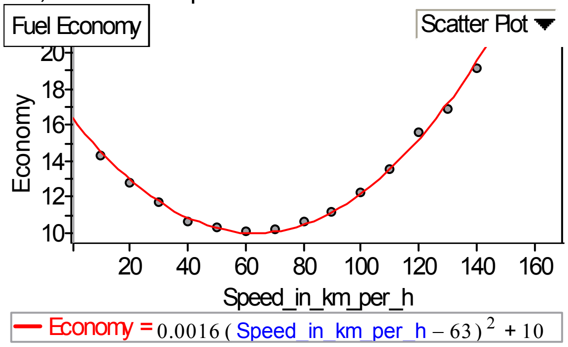
1. **How fast does a rumour spread?**



Identify:	Reasons
Function Type ☐ linear ☐ quadratic ☐ exponential ☐ none of the above	
Domain:	
Range:	

B coaches A

2. The graph models the average fuel economy of a particular car, in litres per 100 km, at various speeds.



(Source: **Mathematics 12: Preparing for College & Apprenticeship**, McGraw-Hill Ryerson, 2002, p. 358)

Identify:	Reasons
Function Type ☐ linear ☐ quadratic ☐ exponential ☐ none of the above	
Domain:	
Range:	

3. The table describes the cooling of a cup of coffee.

Time (min)	0	4	8	12	16	20
Temperature (^{o}C)	55	47	40	34	29	25

Identify:	Reasons
Function Type ☐ linear ☐ quadratic ☐ exponential ☐ none of the above	
Domain:	
Range:	

4. If you invest $5000 in a stock that is increasing in value at a rate of 12% per year, then the value of your stock is given by $A = 5000(1.12)^n$, where *A* is the amount in dollars and *n* is the number of years.

Identify:	Reasons
Function Type ☐ linear ☐ quadratic ☐ exponential ☐ none of the above	
Domain:	
Range:	

5.6.2 Are you the Master of Your Domain and Range? (continued)

Part B: Individual

1. The table below shows that height of a baseball, in metres, after *t* seconds.

Time (s)	Height of Ball (m)
0	0.8
1	25.9
2	41.2
3	46.7
4	42.4
5	28.3

Identify:	Reasons
Function Type ☐ linear ☐ quadratic ☐ exponential ☐ none of the above	
Domain:	
Range:	

2. A computer virus attached to an e-mail can spread rapidly. Once the attachment is opened, the virus will cause an infected e-mail to be sent to everyone in the recipient's address book. Assume that on average, a person has 15 addresses in his or her address book and that people read their e-mail once a day. (Source: **Advanced Functions and Introductory Calculus**, Nelson, 2002, p.93) The following table shows the spread of one computer virus through e-mail over the course of 6 days.

Time (days)	Number of E-mails with Virus
1	15
2	225
3	3 375
4	50 625
5	759 375
6	11 390 625

Identify:	Reasons
Function Type ☐ linear ☐ quadratic ☐ exponential ☐ none of the above	
Domain:	
Range:	

3. A herbicide was sprayed onto a field containing an estimated 5000 weeds. The number of weeds, *N*, still alive after *t* days can be modelled by $N(t) = 5000(0.4)^t$

Identify:	Reasons
Function Type ☐ linear ☐ quadratic ☐ exponential ☐ none of the above	
Domain:	
Range:	

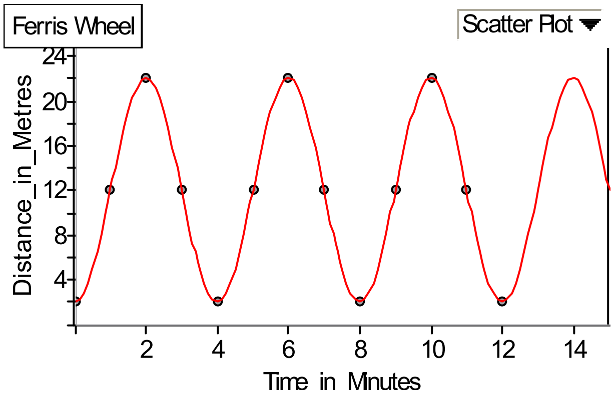
4. $(Pizza)^2$ charges $10 for a large pizza plus $2 per topping. The total cost of the pizza, *C*, can be modelled by $C = 2n + 10$, where *n* is the number of toppings.

Identify:	Reasons
Function Type ☐ linear ☐ quadratic ☐ exponential ☐ none of the above	
Domain:	
Range:	

5.6.2 Are you the Master of Your Domain and Range? (continued)

5. Cheryl is riding a Ferris wheel. The graph below shows Cheryl's height above the ground.

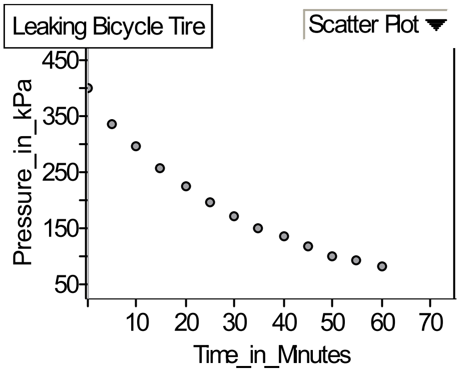
Cheryl's Height vs. Time



Identify:	**Reasons**
Function Type ☐ linear ☐ quadratic ☐ exponential ☐ none of the above	
Domain:	
Range:	

6.

Tire Pressure vs. Time



(Source of data**: Preparing for the new Grade 11 Mathematics: Growth & Change**, Peter Taylor, Queen's University, 1999, p. 105)

Identify:	**Reasons**
Function Type ☐ linear ☐ quadratic ☐ exponential ☐ none of the above	
Domain:	
Range:	

Unit 5: Day 7 : How an Infectious Disease can Spread		**Grade 11 U/C**
Minds On: 10 Action: 40 Consolidate:25 Total=75 min	**Description/Learning Goals** • Identify exponential functions that arise from real world applications involving growth and decay given tables of values, graphs and/or equations. • Determine restrictions that must be placed on the domain and range in order to apply an exponential model. • Collect data that can be modelled as an exponential function from primary sources.	**Materials** • BLM 5.7.1, • BLM 5.7.2 (Teacher) • See BLM 5.7.2 for additional materials

		Assessment Opportunities
Minds On…	**Whole Class → Discussion** Introduce the simulation that will allow students to experience how an "infectious" disease can spread. Explain how students are to interact. Refer to BLM 5.7.2 Teacher Notes for important information related to this activity.	This simulation is adapted from http://serendip.brynmawr.edu/sci_edu/waldron/infectious.html It is important to carefully read the entire activity as well as the Teacher Notes on BLM 5.7.2 before starting this activity.
Action!	**Whole Class → Simulation** Complete steps #1-11 of BLM 5.7.1 **Pairs → Investigation** Complete steps # 12-21 of BLM 5.7.1 **Mathematical Process Focus : Reflecting: (**Students will **reflect** on the reasonableness of an exponential model to fit this data over the whole domain.) **Learning Skill/Teamwork/Observation/Rating Scale:** By observation, assess and record each student's level of teamwork using a four point scale (N, S, G, E).	
Consolidate Debrief	**Whole Class → Discussion** Students share their answers for questions 13-21. Pose the following Extending Questions: a) How many interactions do you estimate you have with other people in a day? b) What implications would this have for the spread of an infectious disease? Add to the FRAME Model by including any additional information.	**Literacy Strategy**: Continue to add to the **Exponential FRAME Model.**
Application Reflection	**Home Activity or Further Classroom Consolidation** Complete question #22 on BLM 5.7.1	

5.7.1 How an Infectious Disease can Spread

An infectious disease is any disease caused by germs that can be spread from one person to another. In this activity, you will simulate the spread of an infectious disease. The activity will demonstrate how one person who is infected with a disease can infect other people who in turn can infect others.

1. Your teacher will give everyone a Dixie cup half-filled with a clear solution. This solution represents a person's body. One person in the class will have a cup that is infected.

2. In this part of the activity you will "interact" with one other student. To "interact" with another student you need to pour all of your solution into his or her Dixie cup. He or she will then pour all of the solution back into your cup. Finally, you pour half of the solution back into his or her cup. You have completed one interaction.

3. Pour several drops of your solution into well #1 of your spot plate. Be careful not to overfill or spill your solution when pouring.

4. How many people do you think will be infected at this point? Place your estimate in the first row of the table below.

Interaction #	**Estimated Number of Infected People**
1	
2	
3	
4	
5	

5. Wait for the signal from your teacher and then move to another part of the classroom and "interact" with a second student.

6. Pour several drops of your solution into well #2 of your spot plate.

7. How many people do you think will be infected at this point? Place your estimate in the next row of the table.

8. Repeat steps 5-7, filling wells #3, 4, and 5. Be sure to WAIT for your teacher's signal before initiating an "interaction".

9. Do you think that you are infected now? ________

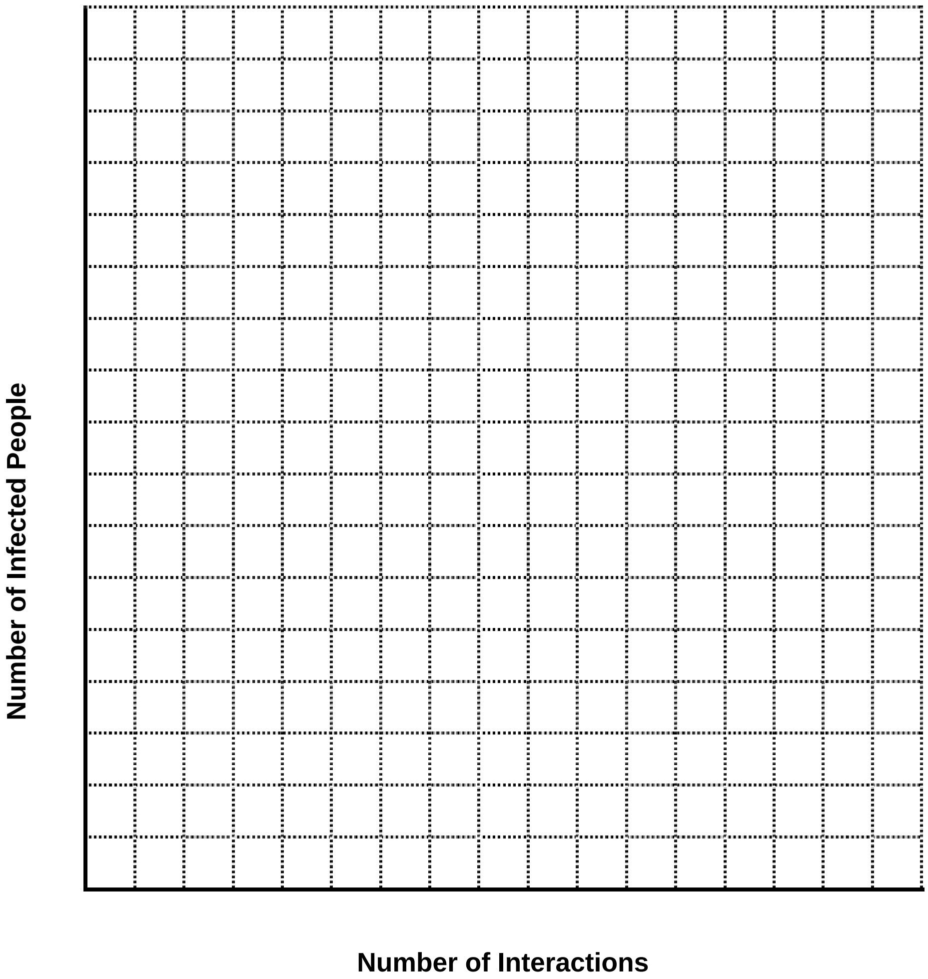
5.7.1 How an Infectious Disease can Spread (continued)

10. Your teacher will add an indicator that will allow you to determine who is infected. When the indicator is added to the solution of an infected person, the solution will turn pink. Together, as a class, complete the following table.

Interaction #	Number of Infected People
0	1
1	
2	
3	
4	
5	

11. Clean up the simulation by properly disposing of the waste. (i.e., Pour all solutions in the designated bucket, place spot plates in the bin and put your empty Dixie cup in the garbage.)

12. Graph the data.



5.7.1 How an Infectious Disease can Spread (continued)

13. This relationship appears to be exponential. Provide reasons to support classifying the relationship as exponential.

14. Using the graph, extrapolate to determine the expected number of infected people after 8 interactions.

15. Examine the pattern in your table of values and the graph. At what rate does the number of infected people appear to increase?

16. Will the disease continue to spread at this rate until everyone in the class is infected? Explain.

17. On the same grid (previous page), graph the function $y = 2^x$, by completing the following table.

x	***y***
0	
1	
2	
3	
4	
5	

5.7.1 How an Infectious Disease can Spread (continued)

18. Compare the graph of the function $y = 2^x$ to the graph representing the spread of infectious disease. Examine all regions of the graphs and identify similarities and differences.

19. At what point does the graph representing the spread of infectious disease no longer fit an exponential function?

20. Over what domain and range does the model representing the spread of infectious disease fit an exponential function?

21. Why does the model representing the spread of infectious disease not follow the pattern of the function $y = 2^x$ for the entire domain of $y = 2^x$?

22. (Complete on another piece of paper.) Refer to BLM 5.6.2, Part B, Question 2. The spread of this e-mail virus could be modelled by the exponential function $y = 15^x$
    a) Can you think of any factors that might lead to restrictions on the domain and range of this model?
    b) Create a graph to compare $y = 15^x$ and the graph that might result from the factors suggested above. Sketch them on the same grid.

5.7.2 Teacher's Notes

Materials

For each student:
- Dixie cup
- spot plate (or 6-well watercolour paint palette)

For the teacher:
- 1 dropper bottle of phenolphthalein
- baking soda
- tap water
- bucket for collecting solutions
- bin for collecting spot plates

Preparing the "Infected" Solution

Dissolve one half of a tablespoon of baking soda in 100 mL of water. The baking soda may not all dissolve. Let the undissolved solid settle and pour off the liquid from the top.

Phenolphthalein Indicator

One dropper bottle of phenolphthalein indicator will be required and may be borrowed from your school's science department (or purchasing information may be obtained from your school's science department.) A 500 mL bottle of phenolphthalein indicator will cost approximately $8.50. You will only need approximately 15 mL to complete this activity with one class.

The paint palette shows that student # 17 became infected after the third interaction.

The teacher adds phenolphthalein to the spot plate for student # 17.

5.7.2 Teacher's Notes (continued)

Prior to the Activity

1. Number each Dixie cup and spot plate from 1 to *n* (where *n* represents the class size). Note: *n* must be an even number. If you do not have an even number of students in your class, give one student two cups and two spot plates and have that student perform two separate "interactions" when the other students perform one "interaction".

2. Number 5 wells on each spot plate with the numbers 1 through 5.

3. Half fill one Dixie cup with the baking soda solution. This solution represents the infected individual.

4. Half fill all the other Dixie cups with tap water.

During the Activity

1. Students must leave each "interaction" with roughly half of the solution in their cup.

2. All students must "interact" with only one other student during each "interaction". Be sure to instruct the class to wait for your signal before starting the next interaction.

3. It is important that students leave their spot plates in one secure location throughout the activity (e.g., designate an infectious disease zone in the room). Students must be careful when pouring solutions into their spot plates or moving around the infectious disease zone so that they do not accidentally contaminate the solution in a well.

4. When all of the "interactions" are done, the teacher should place one drop of phenolphthalein indicator in well #1 of each student's spot plate. When the indicator is added to the solution of an infected person, the solution will turn pink. After the first "interaction" there will be two infected students (the original student and the person they "interacted" with). Have students record the total number of people infected after the first "interaction" in the table on BLM 5.7.1.

5. Continue to use the phenolphthalein indicator to determine the number of infected people after each "interaction". After the second "interaction" two more people (four people in total) should be infected. The pattern continues with the number of infected people doubling. **However**, as the number of "interactions" increases, the probability of two infected people "interacting" increases. When this happens the number of infected people does not exactly double. Therefore, the spread of infection cannot be modelled exactly by the function $y=2^x$ as the number of "interactions" increases. Question #21 of BLM 5.7.1 is referring to this effect and should be included in the discussion when debriefing the activity.

After the Activity: Waste Disposal

Have students pour the solutions from their Dixie cups and the spot plates down the drain of a sink or collect them in a bucket and later dispose the solution down the drain of a sink. The spot plates can be collected in a bin and washed later by the teacher or a student volunteer.

Unit 5 : Day 10 : Using Graphical and Algebraic Models

Grade 11U/C

Minds On:	10
Action:	50
Consolidate:	15
Total=75 min	

Description/Learning Goals

- Identify exponential functions that arise from real-world applications involving growth and decay given graphs and/or equations.
- Solve problems using given graphs or equations of exponential functions based on a variety of real-world applications by interpreting the graphs or by substituting values into the equations.
- Identify key features of graphs of exponential functions (y-intercept, increasing or decreasing, horizontal asymptote, domain and range).
- Determine reasonable restrictions on the domain and range of real-world applications modelled by exponential functions.

Materials

- BLM 5.10.1
- Computer connected to the Internet
- Data Projection Unit

Assessment Opportunities

Minds On…

Whole Class→ Exploration

Run the applet simulating the population growth of a fish habitat. Use both the Habitat and the Graph "views" to show the population grows exponentially through the generations. Use the STEP function to step through each generation one by one and observe the growth.

The simulation begins with 2 fish and the growth rate defaults to 1.5, meaning 50% . Develop the formula with the students to model the growth of this habitat. $y = 2(1.5)^x$. Have students use this algebraic model to calculate the number of fish that would be expected in the 10th and 15th generations. Check the predictions made using the equation, by using the applet.

Change the growth rate in the computer to other values such as 1.3 and 1.7. Observe the changes in the graph. Ask students to consider what the algebraic models would be for these habitats. Help students make connections between the real world example and the algebraic model.

The applet can be found at http://www.otherwise.com/population/exponent.html

Note: Students are not expected to generate exponential equations on their own. However, they should be able to understand the meaning of each term in the exponential equation and relate it to the real-world application.

Action!

Individual → Problem Solving

Students work individually to complete BLM 5.10.1

Mathematical Process Focus: Connecting Students will make **connections** by applying prior knowledge to new contexts. Student will make **connections** between graphical and algebraic models .

Learning Skills (Works Independently)/Observation/Rating Scale: Assess students' ability to work independently.

Consolidate Debrief

Whole Class → Connecting

Review students' answers. Make connections between graphical and algebraic models.

Add to the **FRAME** model by including examples using graphical, algebraic, and contextual models.

Literacy Strategy: Continue to add to the **Exponential FRAME Model.**

Application
Concept Practice
Reflection
Skill Drill

Home Activity or Further Classroom Consolidation

Complete BLM 5.10.1

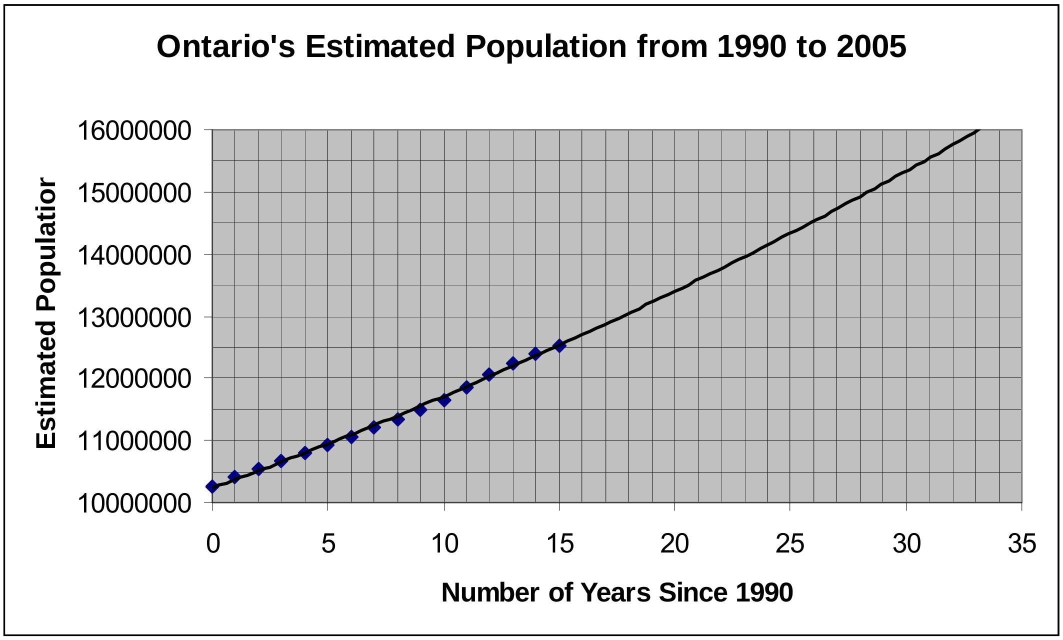
Assign the following problem for homework: The temperature, over time, of a cup of coffee sitting on the counter can be modelled by the exponential function $T(x) = 60\left(\frac{1}{2}\right)^{\frac{x}{30}} + 20$, where $T(x)$ is the temperature, in degrees Celsius, and x is the elapsed time, in minutes. Determine the temperature after 45 minutes, 90 minutes, 2 hours, 2.5 hours and 3 hours. Based on these results, over time, what do you think the temperature of the coffee will become?

Note: When taking up the Home Activity, have students graph the function using the calculator to verify the results and to see the effect of the horizontal asymptote in this problem.

5.10.1 Using Graphical and Algebraic Models

Population Growth

The following graph shows Ontario's estimated population from 1990 to 2005 as found on the Statistics Canada website. The dots represent actual data. A curve of best fit has been added to the graph.



(Source: Statistics Canada)

1. Using the graph, estimate Ontario's population in 1995.

2. Using the graph, predict Ontario's population in 2010. Does this seem reasonable? What are you assuming about the growth pattern?

5.10.1 Using Graphical and Algebraic Models (continued)

3. Ontario's population is growing at a rate of approximately 1.35% per year. The estimated population, *y*, can be modelled by $y = 10269192(1.0135)^t$, where *t* is the number of years since 1990 and 10269192 was the estimated population in 1990.

   a. Use the equation modelling Ontario's population to calculate the estimated population in 1995. Compare this algebraic result to the graphical result from question 1.

   b. Use the equation modelling Ontario's population to calculate the projected population in 2010. Compare this algebraic result to the graphical result from question 2.

   c. When would you use a graphical model to convey mathematical information?

   d. When would you use an algebraic model (i.e., equation) to convey mathematical information?

4. Some people might think that the graph indicates this data is linear. Do you agree? Why or why not?

5. When will Ontario's population reach 15 million? Does this seem reasonable?

6. How would the graph and equation be different if the growth rate was1.5%?

5.10.1 Using Graphical and Algebraic Models (continued)

Rebound Height

When a basketball is correctly inflated, it rebounds to approximately 60% of the height from which it is dropped. A correctly inflated basketball is dropped from a height of 2.4 m and continues to bounce, each time rebounding to 60% of its previous height.

1. The rebound height of the basketball, *h*, can be modelled by the equation $h = 2.4(0.6)^n$, where *n* is the number of rebounds.

    a. Explain the meaning of 2.4 and 0.6 in this equation.

    b. Use the equation to determine the rebound height of the basketball after 5 rebounds.

2. Suppose the ball stops rebounding and begins to roll across the floor when it reaches a rebound height of 3 cm. How many times has the ball rebounded? Explain how you solved this problem.

3. What is the domain and range for the function modelling the rebound height of the basketball?

4. How would the equation change if:

    a. the ball was over-inflated and rebounded to 75% of its previous rebound height.

    b. the ball was dropped from an initial height of 2 m.

5.10.1 Using Graphical and Algebraic Models (continued)

5. Yvonne and Nancy are avid basketball players. After playing basketball outside on a hot summer day, they stop for a lemonade break. Yvonne sits down on the bench of the picnic table while Nancy stands on the bench on the opposite side. Nancy holds the basketball above her head and drops it onto the top of the picnic table from a height of 2.2 m above the top of the picnic table.

   a. Based on the information above, explain why the equation $h = 2.2(0.6)^n$ would model the rebound height of the basketball in relation to the top of the picnic table after *n* rebounds.

   b. The top of the picnic table is 70 cm above the patio. Explain how you think this would affect the rebound height if it is measured from the patio rather than from the top of the picnic table on which it is being bounced.

   c. The equation that models the height of the rebound in relation to the patio is $h = 2.2(0.6)^n + 0.7$.

      i. Calculate the rebound height, <u>in relation to the top of the picnic</u> table on the third bounce.

      ii. Calculate the rebound height, <u>in relation to the patio</u> on the third bounce.

      iii. How do the two values compare?

   d. Yvonne stands on the bench of a different picnic table. She holds the ball over her head and drops the ball onto the table from a height of 1.7 metres above the top of the table. If the top of the table is 80 cm above the patio, suggest how the equation modelling the rebound height in relation to the patio would change.

5.10.1 Using Graphical and Algebraic Models (continued)

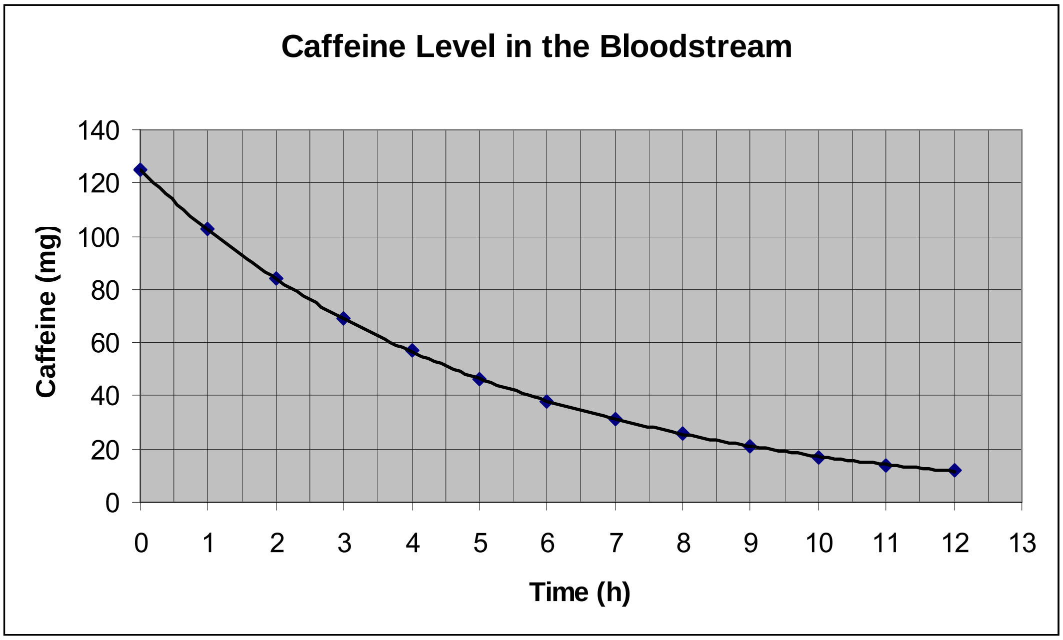
Caffeine Consumption

(Source: **Advanced Functions and Introductory Calculus**, Nelson, 2002, p. 128)

When you drink coffee, tea, or hot chocolate, or eat a chocolate bar, your body absorbs chemicals from these foods, including caffeine. The amount of caffeine in your bloodstream follows an exponential pattern over time.

The highest level of caffeine in the bloodstream occurs 15 min to 45 min after drinking a beverage or eating a food with caffeine. Then the level of caffeine begins to fall.

The following graph shows the caffeine level in the bloodstream of Peter, over time. Peter is an adult smoker who has consumed a cup of coffee (250 mL). The coffee contains 125 mg of caffeine that peaks in his bloodstream shortly after consumption. The graph starts at the time when the caffeine level peaks (i.e., $t = 0$ when the caffeine level peaks).



1. Using the graph, determine the amount of caffeine in Peter’s bloodstream after 4.5 hours.

2. Using the graph, determine when Peter will have 20 mg of caffeine in his system.

5.10.1 Using Graphical and Algebraic Models (continued)

3. The time it takes for half of the original amount of caffeine to remain in the bloodstream is called the half-life. Use the graph to determine the half-life of caffeine in Peter's bloodstream. Show your work on the graph.

The half-life from the graph is: ________________

The length of the half-life of caffeine is affected by a number of factors, including age. The following data shows the half-life of caffeine for a variety of factors.

Factor	Half-life
Adult non-smoker	5.5 h
Adult smoker	3.5 h
Woman who is six months pregnant	10 h to 18 h
Newborn baby	100 h
8-month-old baby	4 h
6-year old to 10-year-old child	2 h to 3 h

An appropriate model for the amount of caffeine in a person's bloodstream is

$$y = c\left(\tfrac{1}{2}\right)^{\frac{t}{h}}, \text{ or } y = c(0.5)^{\frac{t}{h}}$$

where
y is the amount of caffeine in the bloodstream in mg,
c is the initial amount of caffeine in mg,
t is the number of hours since the caffeine level in the bloodstream has peaked
h is the half-life of caffeine in hours (i.e., the amount of time for half of the caffeine to remain in the bloodstream)
0.5 indicates that the caffeine is decaying by a factor of ½ (hence 'half-life')

4. Jenny is an adult and does not smoke. She also consumes a cup of coffee (250 mL). Complete the following table by substituting values into the equation modelling caffeine level.

FIRST

Substitute the values for c and h into the equation $y = c(0.5)^{\frac{t}{h}}$ and write the resulting equation to calculate Jenny's caffeine level at various times (*t=0 to t=12*)

The equation is: ______________

t (hours)	*y* (in mg)
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

5. Graph Jenny's caffeine level over time on the same graph modelling Peter's caffeine level.

6. Compare Jenny and Peter's graphs. In your comparison, discuss the shapes of graphs, the type of functions, the y-intercepts, horizontal asymptotes, domain, and range.

7. Confirm that the half-life of caffeine in Jenny's bloodstream is 5.5 h by using the graph. Show your work on the graph.

8. A woman who is six months pregnant drinks a 250 mL cup of coffee. From the chart, the half-life is 10-18 hours. Assume a half-life of 14 hours. If she doesn't consume any more caffeine, would she have any caffeine left in her bloodstream 2 days later? If so, how much?